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An Economic Variables Sampling Plan with Multi-Decision Alternatives

Do Sun Bai* and Sung Hoon Hong*

多決定 代案을 갖는 經濟的 計量型 샘플링檢査方式

裵道善*·洪性勳*

Abstract

For situations where there are several markets for a product with different profit/cost structures, an economic variables sampling plan is developed for determining the market to ship the lots to. It is assumed that the quality characteristic X is normally distributed with known variability and unknown mean having a normal prior distribution. Profit models are constructed which involve four profit/cost components; profit from a conforming item, inspection cost, replacement cost, and cost from an accepted nonconforming item. Methods of finding optimal sampling plan are presented and a numerical example is given.

1. Introduction

Traditionally, acceptance sampling schemes have largely been based upon statistical considerations; for example, MIL-STD-105D, Dodge-Romig tables, etc. In recent years increased attention has been devoted to acceptance sampling plans based upon economical and/or Bayesian considerations. The design of such sampling plans requires knowledge of the costs of using a given sampling plan, and the specification of prior distribution regarding lot quality. Despite the difficulty of obtaining information about costs and prior distribution, acceptance sampling based on costs and prior distribution has gained much popularity, since the incorporation of costs and prior distribution into the sampling design renders decision making objective and precise. Since 1960's, economic attributes acceptance sampling plans have been considered by many authors; see, for example, Hald(1960), Chiu(1974), Moskowitz, et al.(1979), Riew and Bai(1984), Guenther(1985), Barad(1986), and Tagaras and Lee(1987). There has been, however, relatively little work on the design of economic

* Department of Industrial Engineering, Korea Advanced Institute of Science and Technology

variables acceptance sampling plans. Ailor et al. (1975) considered an economic sampling plan in which the quality characteristic could be a mixture of variables and attributes. Chapman et al. (1978) developed an economic multivariate acceptance sampling plan. Schmidt et al. (1980) developed an economic variables sampling plan based on three decision alternatives : the lots are classified into three quality grades A, B, and C where grade A lots are accepted, grade B lots screened, and grade C lots scraped. Lam(1988) considered an economic variables sampling plan with a polynomial loss function.

In this paper we consider the problem of designing variables sampling plan to grade the quality of the products that can be sold to several markets or used as components of several products with different profit/cost structures. Many industrial examples can be found ; an integrated circuit(IC) can be used as a component of filters, amplifiers, etc., and a relay as a component of tape recorders, clocks, etc. In such situations, a nonconforming IC or relay causes different costs of assembling/disassembling and damage to other parts in the assembly. A conforming one may also yield different profits. It is therefore important to determine the product into which the IC's or relays should be assembled depending on their quality. The decision for disposing the lot is then made based on the quality of the lot estimated by sampling inspection and the profit/cost structure. Economic attributes sampling plans under similar situations were considered by Bai and Hong(1990).

In a sampling inspection, a random sample $X=(X_1, X_2, \cdots, X_n)$ of size n is taken and the decision for disposing the lot is made based on the sample mean $\bar{X}=\Sigma X_i/n$. Methods are developed for determining the sample size n and the set of disposition limits $(\delta_1, \delta_2, \cdots, \delta_m)$ for the markets that maximize the expected profit. Bayesian approach is utilized to define the optimal sampling plan.

The notation and basic assumptions used in this paper are as follows.

Notation

N	= lot size
L	= lower specification limit
n	= sample size
δ_i	= disposition limit for market i, $i=1, 2, \cdots, m$
x	= quality measurement
$\bar{x}$	= sample mean
μ	= mean of x
σ^2	= variance of x
A_i	= profit from a conforming item for market i, $i=1,2,\cdots,m$
$C_i(x, L)$	= cost from an accepted nonconforming item for market i, depending on the quality deviation between x and L, $x<L$, $i=1,2,\cdots,m$
S	= unit inspection cost
D	= unit replacement cost
$f(x \mid \mu)$	= conditional distribution of x given μ
$h(\mu)$	= prior distribution of μ
$g(\bar{x})$	= marginal distribution of $\bar{x}$
$t(x \mid \bar{x})$	= conditional distribution of x given $\bar{x}$
$\ell(\mu \mid \bar{x})$	= posterior distribution of μ given $\bar{x}$
$\phi(\cdot)$	= density function of the standard normal distribution
$\Phi(\cdot)$	= cumulative distribution function of the standard normal distribution

Assumptions

i) There is a lower specification limit L.

ii) The test of sampled item is nondestructive and all nonconforming items found in the sample are replaced by conforming items.

iii) X is normally distributed with unknown mean μ and known variance σ^2.

iv) μ is normally distributed with known mean θ and known variance τ^2.

2. The model

Consider a product that can be sold to M different markets. When an accepted lot is sold to market i, a conforming item yields a profit of A_i and a nonconforming one causes a loss of $C_i(x, L) = a_i(L-x)^2$, depending quadratically on the quality deviation between x and lower specification limit L. A_i may be equal to the difference between the market price and production cost. The cost of accepting a nonconforming item may include service and replacement costs plus loss of goodwill. Now consider markets i and j where $A_i > A_j$ and $C_i(x, L) \leq C_j(x, L)$, or $A_i = A_j$ and $C_i(x, L) < C_j(x, L)$. It is easy to verify that market j is 'dominated' by market i and thus a lot should be sold to market i rather than market j if only these two markets are under consideration. Therefore, we should consider only the markets which are not dominated. Assume that there are m markets which are not dominated. Without loss of generality, we assume that $A_i > A_j$ and $C_i(x, L) > C_j(x, L)$ for all $i < j$. The condition $C_i(x, L) > C_j(x, L)$ for all $i < j$ is equivalent to the condition $a_i > a_j$. If there are too many nonconforming items in the lot, it may be unprofitable to ship the lot to an ordinary market because of the costs caused by accepted nonconforming items. Therefore, we consider market m as an alternative with one of the following modes ; sell the lot at a discount, screen the lot and sell conforming items and scrap nonconforming items, or scrap the lot.

The operating procedure for the sampling plan is as follows :

i) Take a random sample of size n from the lot and compute $\bar{x} = \sum_{i=1}^{n} x_i / n$.

ii) Let δ_i, $i = 1, 2, \cdots, m$, be real numbers such that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_m = -\infty$ and $\delta_0 = \infty$. If $\delta_i \leq \bar{x} < \delta_{i-1}$, $i = 1, 2, \cdots, m$, ship the lot to market i. If $\delta_i = \delta_{i-1}$, the lot is not shipped to market i.

In a sampling inspection, the expected profit per lot will be composed of the following four components.

1) *Profit from conforming items in the accepted lot*

Since all nonconforming items found in the sample are replaced by conforming items, the expected number of conforming items in the accepted lot given $\bar{x}$ becomes

$$n + (N-n) \cdot \int_{-\infty}^{\infty} \int_{L}^{\infty} f(x \mid \mu) \ell(\mu \mid \bar{x}) dx \cdot d\mu$$

$$= n + (N-n) \cdot \int_{L}^{\infty} \pi(x) dx, \quad \cdots\cdots\cdots (1)$$

where $\pi(x) = \int_{-\infty}^{\infty} f(x \mid \mu) \ell(\mu \mid \bar{x}) d\mu$ is a normal density function with mean

$$\mu_\pi = (\sigma^2\theta + n\tau^2\bar{x}) / (n\tau^2 + \sigma^2), \quad \cdots\cdots\cdots (2a)$$

and variance

$$\sigma_\pi^2 = \{1 + \tau^2/(n\tau^2 + \sigma^2)\}\sigma^2. \quad \cdots\cdots\cdots (2b)$$

Thus we obtain the expected profit for market i given $\bar{x}$ as

$$A_i \cdot \{N - (N-n) \cdot \Phi(\xi)\}, \quad \cdots\cdots\cdots (3)$$

where

$$\xi = \frac{L - \mu_\pi}{\sigma_\pi}.$$

2) *Cost from nonconforming items in the accepted lot*

The expected cost from nonconforming items in the accepted lot given $\bar{x}$ becomes

$$(N-n)\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{L} C_i(x,L)f(x\mid\mu)\ell(\mu\mid x)dx\cdot d\mu$$
$$=(N-n)a_i\cdot\int_{-\infty}^{L}(L-x)^2\pi(x)\,dx$$
$$=(N-n)a_i\cdot\sigma_\pi^2\{(1+\xi^2)\Phi(\xi)+\xi\phi(\xi)\}. \quad \cdots (4)$$

3) *Replacement cost*

The expected number of nonconforming items found in the sample depends upon the value of $\bar{x}$. We obtain the expected replacement cost given $\bar{x}$ as

$$nD\cdot\int_{-\infty}^{L}t(x\mid\bar{x})dx=nD\cdot\Phi\left(\frac{L-\bar{x}}{\sigma}\sqrt{\frac{n}{n-1}}\right). \quad \cdots\cdots (5)$$

4) *Inspection cost*

The inspection cost is nS.

By summing formulas (3), (4), (5), and the inspection cost, we obtain the expected profit per lot for market i when sample mean is $\bar{x}$;

$$EP_i(n,\bar{x})=A_i\cdot\{N-(N-n)\cdot\Phi(\xi)\}$$
$$-(N-n)a_i\sigma_\pi^2\{(1+\xi^2)\Phi(\xi)+\xi\phi(\xi)\}$$
$$-nD\Phi\left(\frac{L-\bar{x}}{\sigma}\sqrt{\frac{n}{n-1}}\right)-nS. \quad \cdots\cdots (6)$$

Since a lot is shipped to market i whenever $\delta_i\leq\bar{x}<\delta_{i-1}$, $i=1,2,\cdots,m$, the expected profit per lot is given by

$$K(n,\delta_1,\delta_2,\cdots,\delta_m)=-nS+\sum_{i=1}^{m}\int_{\delta_i}^{\delta_{i-1}}k_i(n,y)g(y)dy, \quad \cdots\cdots (7)$$

where $k_i(n,y)=EP_i(n,y)+nS$ and $g(y)$ is the normal density function with mean θ and variance $(\frac{\sigma^2}{n}+$[illegible]$)$.

3. Optimal sampling plan

The optimal sampling plan can be obtained by maximizing formula (7) with respect to $(n,\delta_1,\delta_2,\cdots,\delta_m)$. We first determine optimal disposition limits $\delta_i^*=\delta_i^*(n)$, $i=1,2,\cdots,m$, for given n and then determine n maximizing the expected profit. For a given n, the expected profit is maximized by choosing the values of $\delta_1,\delta_2,\cdots,\delta_m$ that maximize the second term in formula (7). An upper bound of the second term that we can attain for a given n is $\int_{-\infty}^{\infty}\{\max_i k_i(n,y)\}g(y)\,dy$. This value is clearly attained by shipping the lot to market i whenever $\bar{x}\in I_i$, $i=1,2,\cdots,m$, where I_i is the set of real numbers $\bar{x}$ satisfying the inequalities $k_i(n,\bar{x})\geq k_j(n,\bar{x})$ for all $j\neq i$, simultaneously. Since $k_1(n,\bar{x})-k_j(n,\bar{x})$ for all $j>1$ is an increasing function of $\bar{x}$ for given n (see Appendix), it is clear that I_1 is given by the interval $I_1=[\delta_1^*,\infty)$ where δ_1^* is the smallest real number $\bar{x}$ satisfying the inequalities $k_1(n,\bar{x})\geq k_j(n,\bar{x})$ for all $j>1$, simultaneously. Similarly, if I_i, $i=2,3,\cdots,m$, is not empty, it is of the form $I_i=[\delta_i^*,\delta_{i-1}^*]$ where δ_i^* is the smallest real number $\bar{x}$ satisfying the inequalities $k_i(n,\bar{x})\geq k_j(n,\bar{x})$ for all $j\neq i$, simultaneously. If I_i is empty, we let $\delta_i^*=\delta_{i-1}^*$. The values $(\delta_1^*,\delta_2^*,\cdots,\delta_m^*)$ then satisfy the inequalities $\delta_1^*\geq\delta_2^*\geq\cdots\geq\delta_m^*=-\infty$. These inequalities imply that the decision of which market to ship the lot to is of the ordered type. That is, the larger the sample mean $\bar{x}$, the lower is the index of the market to which the lot is shipped to.

Since $k_i(n,\bar{x})-k_j(n,\bar{x})$ for all $i<j$ is an increasing function of $\bar{x}$, the range of $\bar{x}$ such that $k_i(n,\bar{x})\geq k_j(n,\bar{x})$ is

$$\begin{cases} \bar{x} \geq y_{ij}, & \text{for } i<j, \\ \bar{x} \leq y_{ij}, & \text{for } i>j, \end{cases} \quad \cdots\cdots (8)$$

where y_{ij} is the real number satisfying the equation $k_i(n, \bar{x}) = k_j(n, \bar{x})$. The value of y_{ij} is obtained by

$$y_{ij} = \left(1+\frac{\sigma^2}{n\tau^2}\right)\{L-\sigma_n\xi_2^*\} - \frac{\sigma^2}{n\tau^2}\theta, \quad \cdots\cdots (9)$$

where ξ_2^* is the value of ξ satisfying the equation

$$(1+r+\xi^2)\cdot\Phi(\xi)+\xi\cdot\phi(\xi)=Nr/(N-n), \quad \cdots (10)$$

where

$$r=\left(\frac{A_i-A_j}{b_i-b_j}\right)\left(\frac{1}{\sigma_n^2}\right).$$

In formula (10) it is seen that ξ_2^* is an implicit function of only two variables, namely, r and $N/(N-n)$. Search algorithms such as Newton-Rapson or bisection methods can be used for finding ξ_2^*. For each $i=1,2,\cdots,m$, we determine $\delta_i^* = \delta_i^*(n)$ as the smallest real number satisfying the inequalities (8), simultaneously. After $(\delta_1^*, \delta_2^*, \cdots, \delta_m^*)$ for a given n are determined, the optimal values $(n^*, \delta_1^*, \delta_2^*, \cdots, \delta_m^*)$ can be found by evaluating the expected profit function for each value of n with the corresponding disposition limits, $(\delta_1^*, \delta_2^*, \cdots, \delta_m^*)$. An upper bound for sample size n is obtained by the expected value of perfect information(EVPI) divided by S. EVPI is the expected profit with perfect information about the lot quality minus the expected profit for the optimal disposition of the lot without sampling. That is

$$\mathrm{EVPI}=\int_{-\infty}^{\infty}\{\max_i K_i(N,L\mid\mu)\}h(\mu)d\mu - \max_i\int_{-\infty}^{\infty}K_i(N,L\mid\mu)h(\mu)d\mu,$$

where

$$K_i(N,L\mid\mu)=N\cdot\{A_i\textstyle\int_L^{\infty}f(x\mid\mu)dx - \int_{-\infty}^{L}C_i(x,L)f(x\mid\mu)dx\},$$

which is the profit when a lot with mean μ is shipped to market i without inspection.

We used quadratic cost function as the form of $C_i(x, L)$. We can also consider other forms of $C_i(x, L)$; for instance, a fixed cost function $C_i(x, L) = b_i$, or a linear cost function $C_i(x, L) = c_i(L-x)$. For fixed and linear cost functions, the value of y_{ij} is obtained by formula(9) with ξ_2^* replaced by ξ_0^* and ξ_1^*, respectively, where ξ_0^* and ξ_1^* are the values of ξ satisfying the equations

$$\Phi(\xi)=\frac{N(A_i-A_j)}{(N-n)(A_i+b_i-A_j-b_j)}, \quad \cdots\cdots\cdots (11a)$$

and

$$(s+\xi)\cdot\Phi(\xi)+\phi(\xi)=Ns/(N-n), \quad \cdots\cdots\cdots (11b)$$

respectively, where $s=(A_i-A_j)/\{(c_i-c_j)\cdot\sigma_n\}$.

4. A numerical example

Consider an integrated circuit (IC) that can function accurately if the input impedance X is greater than or equal to 9.0 MΩ. It can be used as a component of a filter or an amplifier. The costs of identifying a nonconforming IC and assembling/disassembling are different. A conforming one also yields different profits. Based on past experiences, it is known that a quadratic cost function is an appropriate one. Profits and acceptance costs in dollars are as follows.

	Amplifier	Filter	Discount
Profit(A_i)	1.8	1.6	0.2
Acceptance cost(a_i)	13.0	7.0	0.0

Lot size, inspection cost and replacement cost are 1000, 1.0, and 4.0, respectively. Inspection history shows that X is $N(\mu, 1.5^2)$, and μ is $N(11, 0.5^2)$. Using these values, we obtain the optimal sampling plan $(n^*, \delta_1^*, \delta_2^*) = (31, 11.71, 10.37)$. Hence IC's are used as amplifier components if $11.71 \leq \bar{x} < \infty$, as filter components if $10.37 \leq \bar{x} < 11.71$, and sold at a discount otherwise. In this case the expected profit is 782.79.

To study the effects of using a wrong form of $C_i(x, L)$, the optimal sampling plans for the fixed and linear cost functions are obtained. The cost coefficients are selected so that the expected profits when a lot is shipped to market i without inspection, $K_i(N, L)$, are exactly the same for the three forms of $C_i(x, L)$, where

$$K_i(N,L) = \int_{-\infty}^{\infty} K_i(N, L \mid \mu) h(\mu) d\mu$$
$$= N\{A_i - (A_i + b_i)\phi(z)\}, \qquad \text{(fixed)}$$
$$= N[A_i - \{A_i + c_i\sqrt{\sigma^2+\tau^2}\, z\}\phi(z) - c_i\sqrt{\sigma^2+\tau^2}\,\phi(z)], \qquad \text{(linear)}$$
$$= N[A_i - \{A_i + a_i(\sigma^2+\tau^2)(1+z^2)\}\phi(z) - a_i(\sigma^2+\tau^2) z\, \phi(z)], \qquad \text{(quadratic)}$$

and

$$z = \frac{L-\theta}{\sqrt{\sigma^2+\tau^2}}.$$

In this example, $b_1 = 12.92$, $b_2 = 6.96$, $b_3 = 0.00$, and $c_1 = 17.16$, $c_2 = 9.24$, $c_3 = 0.00$. Using these cost coefficients, we obtain the optimal sampling plan $(n^*, \delta_1^*, \delta_2^*) = (22, 12.12, 10.22)$ with expected profit 736.30 for the fixed cost function, and $(n^*, \delta_1^*, \delta_2^*) = (27, 11.87, 10.33)$ with expected profit 759.87 for the linear one. In this example there is no significant difference in the optimal sampling plans. This is, however, not always true. The optimal sampling plans and the percentage errors (PE) of the expected profit due to misuse

Table 1. Optimal Sampling Plans and Percentage Errors with Wrong Cost Functions

p	θ	σ^2	τ^2	Optimal Sampling Plans		PE for cost function used (%)		
				True Cost Function	$(n^*, \delta_1^*, \delta_2^*)$	fixed	linear	quadratic
0.10	10.8	1.72	0.25	fixed	(16, 11.53, 9.80)	0.0	0.4	0.9
				linear	(22, 11.31, 9.95)	0.3	0.0	0.1
				quadratic	(25, 11.20, 10.02)	1.0	0.2	0.0
	11.0	2.25	0.25	fixed	(22, 12.12, 10.22)	0.0	0.3	0.8
				linear	(27, 11.87, 10.33)	0.3	0.0	0.1
				quadratic	(31, 11.71, 10.37)	0.9	0.1	0.0
0.15	10.8	2.62	0.40	fixed	(27, 12.61, 10.88)	0.0	0.8	3.1
				linear	(30, 12.28, 10.79)	0.8	0.0	0.6
				quadratic	(33, 12.05, 10.72)	2.4	0.5	0.0
	11.0	3.24	0.49	fixed	(25, 13.25, 11.35)	0.0	2.0	6.3
				linear	(29, 12.84, 11.20)	1.7	0.0	1.0
				quadratic	(32, 12.56, 11.10)	4.7	0.9	0.0
0.20	10.8	3.42	1.21	fixed	(17, 13.59, 11.86)	0.0	8.7	29.5
				linear	(21, 13.03, 11.50)	5.8	0.0	4.1
				quadratic	(24, 12.65, 11.26)	14.5	3.2	0.0
	11.0	4.12	1.44	fixed	(16, 14.23, 12.36)	0.0	10.0	34.4
				linear	(20, 13.60, 11.93)	6.4	0.0	4.9
				quadratic	(23, 13.17, 11.65)	16.1	3.6	0.0

of the cost function are given in Table 1 for selected combinations of θ, σ^2, and τ^2 with the remaining parameters fixed. The parameters θ, σ^2, and τ^2 and chosen so that the process average $p=P(X<L)=\phi\{(L-\theta)/(\sigma^2+\tau^2)^{1/2}\}$ is 0.10, 0.15, and 0.20. PE is expressed as

$$PE=\frac{EP^*-EP'}{EP^*}\times 100(\%), \quad \cdots\cdots\cdots\cdots\cdots\cdots (12)$$

where EP^* and EP' are the expected profits by using the correct cost function and a wrong cost function, respectively. Table 1 shows that, for large process averages, using the correct cost function is important in implementing a sampling plan. The profit decrease is as high as 34.5% for process average 0.20.

5. Concluding remarks

We have developed an economic variables sampling plan for grading the quality of the products that can be sold to several markets or used as components of several products with different profit/cost structures. The profit/cost components considered in the model are profit from a conforming item, inspection cost, replacement cost, and cost from an accepted nonconforming item which is a function of the deviation of x from L. Three forms of cost functions—fixed, linear, and quadratic—are considered. Empirical results show that, for large process averages, a correct use of cost function is important in designing a sampling plan.

Profit models are constructed under the assmption that all nonconforming items found in the sample are replaced by conforming items. The models can easily be modified to accomodate other modes of disposing the sampled items; for instance, all sampled items may be put back to the lot or nonconforming items found in the sample may be eliminated. The models can also be extended to the products where there are both lower and upper specification limits. It is sometimes difficult to obtain accurate profit and cost information and it will be of interest to study how sensitive this model is to the changes of profit/cost factors.

References

[1] Ailor, R.H., Schmidt, J.W. and Bennett, G.K., "Design of Economic Acceptance Sampling Plans for a Mixture of Variables and Attributes," *AIIE Transactions*, Vol. 7, pp.370−378, 1975.

[2] Bai, D.S., and Hong, S.H., "Economic Design of Sampling Plans with Multi−Decision Alternatives," *Naval Research Logistics*, Vol. 37, pp. 905−918, 1990.

[3] Barad, M., "Using Break−Even Quality Level for Selecting Acceptance Sampling Plans Given a Prior Distribution," *International Journal of Production Research*, Vol. 24, pp.65−72, 1986.

[4] Chapman, S.C., Schmidt, J.W. and Bennett, G.K., "The Optimum Design of Multivariate Acceptance Sampling Plans," *Naval Research Logistics Quarterly*, Vol. 25, pp.633−651, 1978.

[5] Chiu, W.K., "A New Prior Distribution for Attributes Sampling," *Technometrics*, Vol. 16, pp. 93−102, 1974.

[6] Guenther, W.C., "Rectifying Inspection for Nonconforming Items and the Hald Linear Cost Model," *Journal of Quality Technology*, Vol. 17, pp.81−85, 1985.

[7] Hald, A., "The Compound Hypergeometric Distribution and a System of Single Sampling Inspection Plans based on Prior Distributions and Co-

sts," *Technometrics*, Vol. 2, pp. 275−340, 1960.

[8] Lam, Y., "Bayesian Approach to Single Variable Sampling Plans," *Biometrika*, Vol. 75, pp. 387−391, 1988.

[9] Moskowitz, H., Ravindran, A., and Patton, J. M., "An Algorithm for Selecting an Optimal Acceptance Plan in Quality Control and Auditing," *International Journal of Production Research*, Vol. 17, pp. 581−594, 1979.

[10] Riew, M. C., and Bai, D. S., "An Economic Attributes Acceptance Sampling Plan with Three Decision Criteria," *Journal of Quality Technology*, Vol. 16, pp. 136−143, 1984.

[11] Schmidt, J. W., Bennett, G. K., and Case, K. E., "A Three Action Cost Model for Acceptance Sampling by Variables," *Journal of Quality Technology*, Vol. 12, pp. 10−18, 1980.

[12] Tagaras, G., and Lee, H. L., "Optimal Bayesian Single−Sampling Attribute Plans with Modified Beta Prior Distribution," *Naval Research Logistics Quarterly*, Vol. 34, pp. 789−801, 1987.

APPENDIX

Proof that $k_i(n,y)-k_j(n,y)$ for $i<j$ is an increasing function of x for given n

$$k_i(n,y)-k_j(n,y)=(A_i-A_j)\cdot\{N-(N-n)\cdot\Phi(\xi)\} -(N-n)(a_i-a_j)\sigma_\pi^2\{(1+\xi^2)\Phi(\xi) +\xi\cdot\phi(\xi)\},$$

from which we obtain

$$\frac{\partial}{\partial y}\{k_i(n,y)-k_j(n,y)\}=(N-n)R[(A_i-A_j) \cdot\phi(\xi)+2(a_i-a_j)\sigma_\pi^2\{\Phi(\xi) +\xi\phi(\xi)\}], \quad \cdots\cdots (A1)$$

where

$$R=\frac{1}{\sigma}[(1+\frac{\sigma^2}{n\tau^2})\{1+\frac{\sigma^2+\tau^2}{n\tau^2}\}]^{-1/2}.$$

Since $\phi(\xi)+\xi\cdot\Phi(\xi)=\int_{-\infty}^{L}(L-x)\pi(x)dx/\sigma_\pi>0$, $A_i>A_j$, and $a_i>a_j$ for $i<j$, formula (A1) is positive and therefore $k_i(n,y)-k_j(n,y)$ is an increasing function of y for given n.

Q. E. D.