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〈事例研究〉

The Exponentially Weighted Moving Average Control Charts

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ABSTRACT

The null hypothesis being tested by the $\bar{X}$ control chart is that the process is in control at a quality level μo.An $\bar{X}$control chart is a tool for detecting process average changes due to assingnable causes.

The major weakness of the $\bar{X}$ control chart is that it is relatively insensitive to small changes in the population mean.

This paper presents one way to remedy this weakness is to allow each plotted value to depend not only on the most recent subgroup average but on some of the other subgroup averages as well. Two approaches for doing this are based on (1) moving averages and (2) exponentially weighted moving averages of forecasting method.

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1. Introduction

Almost every manufacturing process results in some random variation in the items it produces. That is, no matter how stringently the process is being controlled, there is always going to be some variation between the items produced.

This variation is called chance variation and is considered to the process. However, there is another type of variation that sometimes appears. This variation, for from being inherent to the process, is due to some assignable cause and usually results in an adverse effect on the quality of the items produced. For instant, this latter variation may be caused by a faulty machine setting, or by poor quality of the raw materials presently being used, or incorrect software, or human error, or any other of a large number of possibilities.

The determination of whether a process is in or out of control is greatly facilitated by the use of control charts.

All control charts provide a rational basis for determining when to take corrective action on the process. Essentially, the control charts minimize two kinds of decision errors: (1) looking for trouble that does not exist, and (2) failing to look for trouble that does exist.

The major weakness of $\bar{X}$control chart is that it is relatively insensitive to small changes in the population mean. That is, when such a change occurs, since each plotted value is based on only a single subgroup and so tends to have a relatively large variance, it takes, on average, a large number of plotted values to detected the change. One way to remedy this weakness is to allow each plotted value to depend not only on the most recent subgroup average but on some of the other subgroup average as well.

Two approaches for doing this are based on (1) moving averages and (2) exponentially weighted moving averages.

2. Moving Average Control Chart

The moving average control chart of span size K is obtained by continually plotting the average of the K most recent subgroups. That is, the moving average at time t, call it M_t. is definded by

$$M_t = \frac{[\, \bar{X}_t + \bar{X}_{t-1} + \cdots + \bar{X}_{t-k+1} \,]}{k} \qquad (1)$$

The successive computations can be easily performed by nothing that

$$kM_t = \bar{X}_t + \bar{X}_{t-1} + \cdots + \bar{X}_{t-k+1} \tag{2}$$

and substituting t+1 for t

$$kM_{t+1} = \bar{X}_{t+1} + \bar{X}_t + \cdots + \bar{X}_{t-1+2} \tag{3}$$

subtraction now yields that

$$M_{t+1} = M_t + \frac{[\bar{X}_{t+1} - \bar{X}_{t-k+1}]}{k} \tag{4}$$

$$M_t = \frac{[\bar{X}_1 + \cdots + \bar{X}_t]}{k} \quad \text{if } t < k \tag{5}$$

$$E[M_t] = \mu$$

$$V(M_t) = \begin{cases} \sigma^2/nt & \text{if } t < k \\ \sigma^2/nk & \text{otherwise} \end{cases} \tag{6}$$

$$UCL = \begin{cases} \mu + 3\sigma/\sqrt{nt} & \text{if } t < k \\ \mu + 3\sigma/\sqrt{nk} & \text{otherwise} \end{cases}$$

$$LCL = \begin{cases} \mu - 3\sigma/\sqrt{nt} & \text{if } t < k \\ \mu - 3\sigma/\sqrt{nk} & \text{otherwise} \end{cases} \tag{7}$$

Example 1. I considered a population that, when in control, is known to be normal with mean 10 and variance $\sigma^2=4$. I supposed that there was a change in the mean value from 10 to 11 (an increase of5σ) and then simulated 25 subgroup average (of size n=5) from this change distribution. That is, I simulated the value of $\bar{X}t$, t=1, 2,⋯,25 that are normally distributed with mean 11 and variance 4/5. Table 1 presents the 25 value along with the moving average based on span size k=8 as well as the upper and lower control limits.

Table 1. the 25 value along with the moving average based on sample size k=8

t	$\bar{X}_t$	M_t	LCL	UCL
1	9.617728	9.617728	7.316719	12.68328
2	10.25437	9.936049	8.102634	11.89737
3	9.876195	9.913098	8.450807	11.54919
4	10.79338	10.13317	8.658359	11.34164
5	10.60699	10.22793	8.8	11.2

6	10.48396	10.2706	8.904554	11.09545
7	13.33961	10.70903	8.95815	11.01419
8	9.462969	10.55328	9.051318	10.94868
9	10.14556	10.61926		
10	11.66342	10.79539		
*11	11.55484	11.00634		
*12	11.26203	11.06492		
*13	12.31473	11.27839		
*14	9.220009	11.1204		
15	11.25206	10.85945		
*16	10.48662	10.98741		
17	9.025091	10.84735		
18	9.693386	10.6011		
19	11.45989	10.58923		
20	12.44213	10.73674		
21	11.18981	10.59613		
22	11.56674	10.88947		
23	9.869849	10.71669		
24	12.11311	10.92		
*25	11.48556	11.22768		

* =Out of control.

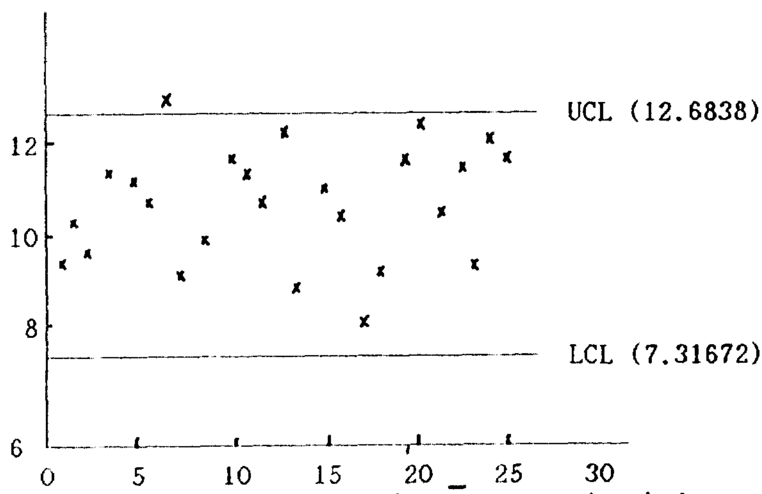

Figure 1.Control chart for $\bar{X}$; span size k=1

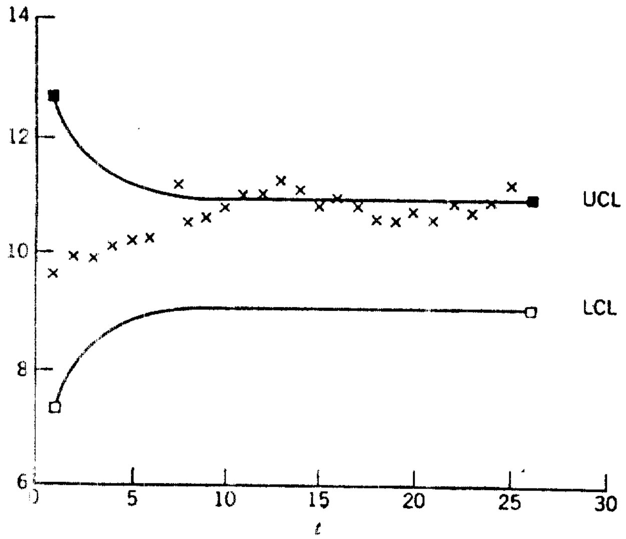

Figure 2. Moving average control chart

3. Exponentially Weighted Moving Average Control Chart

Exponential smoothing is used extensively in the forecasting field. In particular, it enjoys wide spread usage for forecating demands for use in inventory control.

Exponential smoothing may be considered a simple from of proportion control since it weights new information with past information in order to produce a new estimate or forecast. For control chart usage, exponential weigh is used in a slightly different form.

$$W_t = \alpha \overline{X}_t + (1-\alpha)\ W_{t-1} \tag{8}$$

Where W_t: the exponentially weighted moving average at the present time, t, and $W_1 = \overline{X}_1$

W_{t-1}: the weighted moving average at the immediately preceeding time period.

$\overline{X}_t$: the present observation

α: the weighting factor for the present observation $0 < \alpha < 1$

where $W_0 = \mu$

$$\begin{aligned} W_t &= \alpha \overline{X}_t + (1-\alpha)\ [\alpha\ \overline{X}_{t-1} - (1-\alpha)\ W_{t-2}] \\ &= \alpha \overline{X}_t + \ \alpha(1-\alpha)\ \overline{X}_{t-1} + (1-\alpha)^2\ W_{t-2} \\ &= \alpha \overline{X}_t + \ \alpha(1-\alpha)\overline{X}_{t-1} + \alpha(1-\alpha)^2\ [\alpha \overline{X}_{t-2} + (1-\alpha)\ W_{t-3}] \\ &= \ \alpha \overline{X}_t + \alpha(1-\alpha)\overline{X}_{t-1} + (1-\alpha)^2\ \overline{X}_{t-2} + (1-\alpha)^3\ W_{t-3} \end{aligned}$$

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$$=\alpha X_t+\alpha(1-\alpha)\bar{X}_{t-1}+\alpha(1-\alpha)^2\bar{X}_{t-2}+\cdots+\alpha(1-\alpha)^{t-1}\ \bar{X}_1+(1-\alpha)^t\mu \quad (9)$$

$$\alpha(1-\alpha)^{i-1}=\bar{\alpha}e-\beta^i$$

$$\bar{\alpha}=\frac{\alpha}{(1-\alpha)},\quad \beta=-\log(1-\alpha) \quad (10)$$

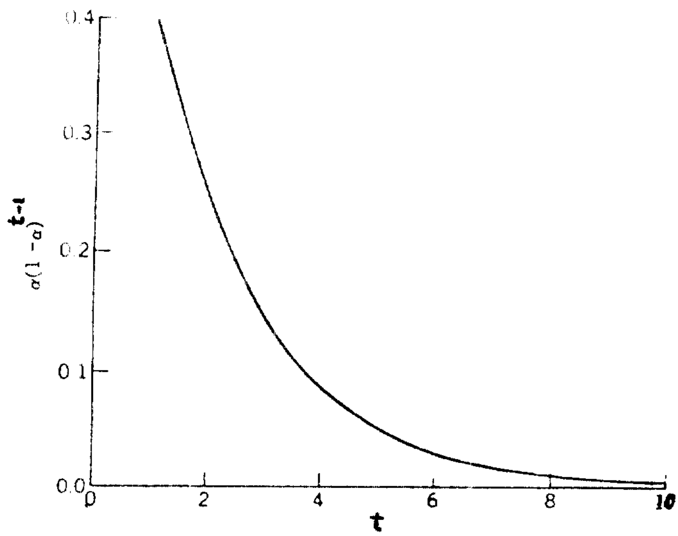

Figure 3. Plot of $\alpha(1-\alpha)^{t-1}$ when α = 0.4

$$E[\hat{W}_t]=\mu[\alpha+\alpha(1-\alpha)+(1-\alpha)^2+\cdots+(1-\alpha)^{t-1}+(1-\alpha)^t]$$

$$=\frac{\mu\alpha\ [1-(1-\alpha)^t]}{1-(1-\alpha)}+\mu(1-\alpha)^t \quad (11)$$

$$=\mu$$

$$V(W_t)=\sigma^2/n\{\alpha^2+[\alpha\ (1-\alpha)]^2+[(1-\alpha)^2]^2+\cdots+\ [\alpha\ (1-\alpha)]^{t-1}\}^2$$

$$=\alpha^2/n\ \alpha^2\ [1+\beta+\beta^2+\cdots+\beta^{t-1}]$$

where $\beta=(1-\alpha)^2$

$$=\frac{\sigma^2\alpha^2[1-(1-\alpha)^{2t}]}{n[1-(1-\alpha)^2]}$$

$$=\frac{\sigma^2\alpha\,[1-(1-\alpha)^{2t}]}{n(2-\alpha)} \quad (12)$$

$$E[W_t]=\mu$$

$$V(W_t) = \frac{\sigma^2\alpha}{n(2-\alpha)} \quad \text{since}(1-\alpha)^{2t}=0 \quad (13)$$

$$UCL=\mu+3\sigma\sqrt{\frac{\alpha}{n(2-\alpha)}}$$

$$LCL=\mu-3\sigma\sqrt{\frac{\alpha}{n(2-\alpha)}} \quad (14)$$

$$\frac{3\sigma}{\sqrt{nk}}=3\sigma\sqrt{\frac{\alpha}{n\,(2-\alpha)}} \quad (15)$$

$$k=\frac{2-\alpha}{\alpha} \text{ or } \alpha=\frac{2}{k+1} \quad (16)$$

Example 2 Consider the data of Example 1 but an exponentially weighted moving average control chart with $\alpha=2/9$

Table 2. the 25 value along with exponentially weights with $\alpha=2/9$

t	$\overline{X}_t$	W_t	t	$\overline{X}_t$	W_t
1	9.617728	9.915051	14	9.220009	10.84522
2	10.25437	9.990456	15	11.25206	10.93563
3	9.867195	9.963064	16	10.48662	10.83585
4	10.79338	10.14758	17	9.025091	10.43346
5	10.60699	10.24967	18	9.693386	10.269
6	10.48396	10.30174	19	11.45989	10.53364
7	13.33961	10.97682	*20	12.44213	10.95775
8	9.462969	10.64041	*21	11.18981	11.00932
9	10.14556	10.53044	*22	11.56674	11.13319
10	11.66342	10.78221	23	9.869849	10.85245
*11	11.55484	10.95391	*24	12.11311	11.13259
*12	11.26203	11.02238	*25	11.48656	11.21125
*13	12.31473	11.20957			

* =Out of control

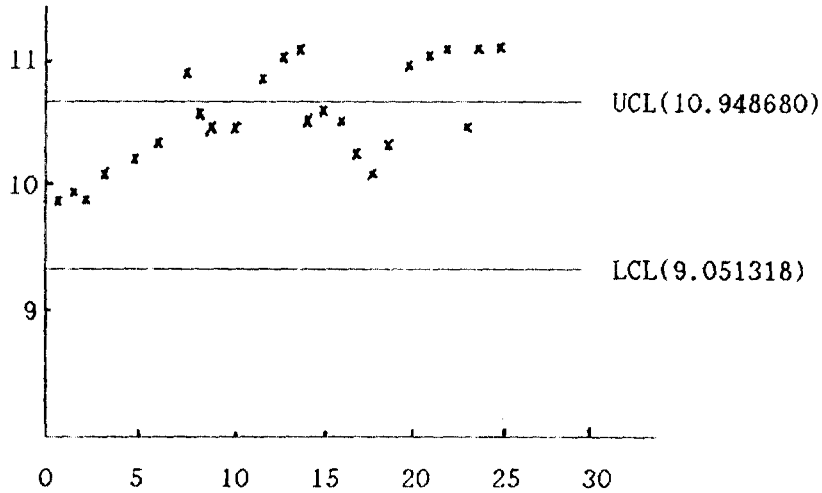

Figure 4. Exponentially weighted moving average control chart with $\alpha = 2/9$

4. Conclusion

$\bar{X}$control chart —that is, the moving average with k=1 — would have declared the process out of control at time 7.

Moving average control chart with k=8 cna see to fall outside its control limits occurred t time 11, 12, 13, 14, 16, 25. EWMA control chart can see out of control at time, 7, 11, 12, 13, 20, 22, 24, 25.

EWMA control chart can not improve on the $\bar{X}$ control chart for detecting relatively large shifts, the EWMA control chart provides greater sensitivity to the change.

For $\alpha=1$, the EWMA places all of its weight on the most recent observation, as does $\bar{X}$ contro chart.

For α close to zero, the most recent observation receives little weight, The choice of between zero and one determines how much weight the most recent observation will receive.

To design an EWMA control chart, constant and appropriate control limits must be specified.

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