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# Optimum Inspection and Replacement Policy in Redundant System

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## Abstract

In this paper, an inspection and replacement policy in a redundant system is considered. It is assumed that the state of the redundant system is known by inspection. When the system is inspected, it is preventively replaced only if the number of failed units exceeds the predetermined limit. Otherwise, the system is inspected after a inspection interval which depends on the number of failed units. We obtain the optimal number of redundant units, inspection intervals and replacement limit minimizing the expected cost rate.

## I . Introduction

In this paper, a situation in which both decisions in design phase and system monitoring in operational

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phase are simultaneously controlled. There are few study related to this topic. Nakagawa [2,3] discussed a joint design and control problems:optimal designs of redundant systems with preventive replacement based on the system age. However, the preventive replacement depends only on the system age in Nakagawa[2,3]. Therefore, when the state of the system(the degraded state) can be known by inspection further cost savings may be possible if preventive replacement decisions are based on the information of inspection instead of simple system age.

In this paper, we study a cost limit replacement polcy in redundant system with common mode failures(Cmfs). The system consist of identical redundant units. It is assumed that the state of the system is known only by inspection. When the system is inspected, it is preventively replaced only if the number of failed units exceeds the predetermined limit(control limit rule[1,4]). Otherwise, the preventive replacement is delayed to next inspection time of which depends on the state of the system. The expected cost rate is used as criterion. we obtain the optimal inspection intervals and replacement limit minimizing the expected cost rate.

**Assumptions**

1. Cmfs and random failures are independant.
2. Failure times of all units are i.i.d.
3. The conditions of system except system failures are detected only by inspection.
4. Inspection and replacement time are negligible.
5. Hazard rate of random failures and Cmfs are constant.
6. Planning horizon is infinite.

**Notation**

| | | |
|---|---|---|
| $n$ | : | redundant number of units in a system |
| $i,j$ | : | index for the number of failed units in a redundant system |
| $S_i$ | : | state of the system in which i units fail $i=0,...,n$ |
| $S_m$ | : | set of states in which system fails |
| $\lambda$ | : | hazard rate of units |
| $\alpha$ | : | hazard rate of Cmfs |
| $\beta_i$ | : | transition rate from state $S_i$ to $S_{i+1}$ |
| $\lambda_i$ | : | transition rate from state $S_i$ to other sate, $\alpha+\beta_i$ |
| $P_{ij}(t)$ | : | Pr{system is in $S_j$ in at t \| system was in $S_i$ at 0} |
| $p_{ij}(t)$ | : | $dP_{ij}(t)/dt$ |
| $P_i(t)$ | : | $P_{i}(t)$ |

$T_i$ : time to reach system failure from state $S_i$
$F_i(t)$, $f_i(t)$, $r_i(t)$ : Cdf, pdf, hazard rate of $T_i$
$\bar{F}_i(t)$ : $1-F_i(t)$

Now we obtain the relations of the various verious terms for the three well-known system structrures. For parallel system with n identical units(the hazard rate of a units is $\lambda$), $S_m=\{S_n\}$and $\beta_i$ $(n-i)\lambda$, $i=0,...,n$. For cold standby system with n identical units(the hazard rate is $\lambda$), $S_m=\{S_n\}$ and $\beta_i$ $=\lambda$ for all i. For k out of n system with n identical units(the hazard rate is $\lambda$), $S_m=\{S_i \cdot i>n-k\}$ and $\beta_i$ $=(n-i)\lambda$, $i=0,...,n-k$.

We now treat the transition probability of the system state with neither inspection nor replacement (refer[1]).
Representation of $P_{ij}(t)$:

$P_{ij}(t)=0$, for $i>j$

$P_{ij}(t)=\beta_i\beta_{i+1}...\beta_j[e^{-\lambda}k^t\bigcap_{ki}]$ for $i<j$

where $\bigcap_{ki}=\prod_{l=1,l\neq k}^{j}(\lambda_l-\lambda_k)$.

$$P_i(t)=\exp(-\lambda_i t) \quad (1)$$

$$P_{ij}(t)\int_0^t \beta_i e^{-\lambda_i T} P(t-T)d\tau, \quad \text{for } 0<i<j<n \quad (2)$$

$$F_{ij}(t)=e^{-\lambda t}+\int_0^t e^{-i\tau}F_{i+1}(t-\tau)d\tau \quad (3)$$

From(1)-(3), we have

$p_{ij}(t)=-\lambda_i P_{ij}(t)+\beta_i P_{i+1}j(t)$, for$0<i<j<n$

$f_i(t)=\lambda_i F_i(t)-\beta_i F_{i+1}(t)$

## II. The Model

In this section, we show that our problem is formulated by a semi-markov decision process in which the number of system state is also a decision vaviable.

### Additional Notation

$E_0$ : event that the system is in state $S_0$

$E_i$ : event that the system is in state $S_i$ when inspection have just been performed.

$E_m$ : event that the system is in state $S_m$

$D_i$ : the decision to be selected when the system has been observed by inspection to be $S_i$

$c_1$ : inspection cost

$c_2$ : unit cost

$c_3$ : additional cost for preventive replacement

$c_4$ : additional cost for corrective replacement

$c_{pr}$ : total cost for preventive replacement, $c_1 n + c_3$

$c_{cr}$ : total cost for corrective replacement, $c_1 n + c_4$

$g$ : expected cost rate for a given policy

$v_i$ : relative value for state $E_i$ for a given policy($v_0=0$)

### Replacement policy:

When the system is observed by inspection to be in state E(0,1,...,m), only one of the two can be made:

PR : the system is preventively replaced

I(t) : the system will be inspected after t,t>0

For redundant systems, $E_0$, $E_1$,..., $E_m$ constitute a semi-Markov process and that the process has a single imbedded markov chain which is ergodic for every stationary policy(see[1]).

Expected time to the next transition is

$$T_i(t) = \begin{cases} \int_0^t \bar{F}_i(x)dx, & D_i = I(t), \\ 0, & D_i = PR \end{cases}$$

Expected cost to the next transition is

$$C_i(t) = \begin{cases} c_{cr}F_i(t) + c_1\bar{F}_i(t) & D_i = I(t), \\ c_{pr}, & D_i = PR \end{cases}$$

1 − step transition probability is −

1) for $D_i = I(t)$, $P_{ij}(t)$ to state $E_i(j=1,...,m)$ for $F_i(t)$ to state $E_0$,

2) for $D_i = PR$, 0 to state $E_i(i=1,...,m)$ and and 1 to state $E_0$

The following definitions are introudced to use for obtaining the optimal policy.

$$V_i = \begin{cases} [c_i(t) + \sum_{j=i+1}^{m} P_{ij}(t) v_j]/P_i(t), & \text{for } D_i = I(t) \\ c_{pr}, & \text{for, } D_i = PR \end{cases} \quad (4)$$

$$W_i = \begin{cases} T_i(t)/P_i(t), & \text{for } D_i = I(t) \\ 0, & \text{for, } D_i = PR \end{cases} \quad (5)$$

Then, from the theory of semi − Markov decision process, the expected cost rate and relative values are the solution of the following equations.

$$gw_i + v_i = V_i, \; i = 1,...,m$$
$$gw_0 = V_0 \quad (6)$$

New notations useful in the policy improvement routine are introduced,

$$V_i(u) = \begin{cases} [c_i(t) + \sum_{j=i+1}^{m} P_{ij}(t) v_j(u)]/P_i(t), & \text{for } D_i = I(t) \\ c_{pr}, & \text{for, } D_i = PR \end{cases} \quad (7)$$

$$v_i(u) = \quad V_j(u) - u_i W_i. \quad (8)$$

$v^*_i(u)$ is the minimum of $v_i(u)$ over all policies for each i, for a given u.

Hence, $v_i(u)$ and $V_i(u)$ are to be obtained when a value of u and a policy are given.

$v_i(u)$ can be obtained by the procedure:

First, find the decision which minimize $v_m(u)$, then find the decision which minimizes replacing $v_{m-1}(u)$ after replacing $v_m(u)$ by $v^*_m(u)$ and so on.

## ALGORITHM

Input data : system structure, $c_1, c_2, c_3, c_4$,

Step 1 : Do n = 1, N(N is the physical constraint).

Step 2 : Guess an initial policy and $g^* = \infty$

Step 3 : For the current policy, solve (6) for g and $v_i$, If $g<g^*$, $g^*=g$
Otherwise the optimal policy for a given n has been obtained, and then go to Sptep 1 ; otherwise, go to next step.
Step 4 : Usin g obtained in Step 3, find the policy which nimimze $v_0(g)$. This policy can be constructed by the procedure obtaining the policy which gives $v_i(g)$. Go to Step 3.

## Ⅲ. Properties of Optimal Policy

In this section, we derive some theorems which are useful to find the optimal policy.

Additional notation

$\mu$ : $F\{T_i\}=\int_0^\infty \bar{F}(t)dt$

$H_i(t,g)$ : $[c_i(t)-gT_i(t)+\sum_{j=i+1}^{m} P_{ij}(t)v_j(g)]/P_i(t)$

$D_i(g)$ : the decision which gives $v_i(g)$

$t^*_i(g)$ : the value of t which minimizes $H_i(t,g)$

$v_i^*(g)$ : $\min\{H(t_i(g), g), c_{pr}\}$

$D_i^*$ : the optimal decision for state $E_i$

Theorem 1 : $D_{m-1}(g)=I(\infty)$or PR
Proof. we know easily that

$$F_{m-1}(t)=1-\exp(-\lambda t),$$
$$T_{m-1}(t)=[1-\exp(-\lambda_i t)]/\lambda_i \text{ and } P_{m-1}(t)=F_{m-1}(t)$$

Hence,

$$H_{m-1}(t, g)=[c_{cr}F_{m-1}(t)+c_1\bar{F}_{m-1}(t)-gF_{m-1}(t)/\lambda_i+F_{m-1}(t)c_{pr}]/F_{m-1}(t)$$
$$=c_{cr}+c_{pr}-g/\lambda_i+c_1\bar{F}_{m-1}(t)/F_{m-1}(t)$$

Is decreasing in t. $H_{m-1}(\infty,g)=c_{cr}-g/\lambda_i+c_{pr}$Q.E.D.

Theorem 2 : If $D_i(g)=PR$, then for all $j>i$, $D_j(g)=PR$. This theorem implies that the optimal policy is the control limit replacement.
Proof. Suppose that there exist k, l($k<l$) such that

$D_k(g) \neq PR,$ (9)

$D_l(g) \neq PR$ (10)

$D_j(g) = PR,$ for $l+1<j<m$ (11)

we derive a contradiction

From(9) we have for all $t(>0)$, $c_{pr} < H_k(t,g)$ and then

$\{1-P_{kk}(t)\}c_{pr} \leq H_k(t,g)\{1-P_{kk}(t)\}$ (12)

Moreover,

$\{-P_{kk}(t)\}H_k(t,g) < [C_k(t) - gTi(t) + Pij(t)Cpr]$ (13)

hold by using $P_{km}(t) > 0$, $v(l) < c_{pr}$.

Hence, we obtain by (12) and (13) that for all $t(>0)$

$$C_K(t) - gT_i(t) + \sum_{j=k}^{m} P_{Kj}(t)C_{pr} - C_{pr} = C_{cr}F_k(t) + C_l F_K(t) - g\int_0^T \overline{F_i}(t)dt$$

$$+ \sum_{j=k}^{m} P_{kj}(t)C_{pr} - C_{pr} = \{C_{cr} + C_l - C_{cr}\}F_k(t) - g\int_0^T \overline{F_K}(t)dt - C_{pr} + C_{cr} > 0$$

Let l.h.s. of the above inequality be $Q_k(t)$. If $c_{pr}+c_l-c_{cr}<0$, $Q_k(t)$ is increasing in k. Hence $Q_l(t)>0$ and $H_l(g,t)>c_{pr}$ for all $t>0$.

and this contradicts(10). Q.E.D.

This theorem implies that he optimal policy is the control limit replacement.

Theorem 3 : $H_i(t,g)$ has at most one minimum with respect to t.

Proof. $dH_i(t)/dt = \sum_{j=l}^{m} P_{ij}(t)h_j$, where $h_j = \lambda_i v_j(g) + v_{j+1}(g) + (c_{cr} - c_0) - g^0$. We note the from theroem 2, there exists $k(0<k<m)$ satisfying

$D(i) \neq PR$ for $0<i<k$

$D(i) = PR$ for $k<i<m$

Where $0<i<k$. Since $I(t_i)$ is optimal inspection interval,

$0 = dH_i(t)/dt = -\lambda_i v_i(g) + \beta_i H_{i+1}(t) - g^0 + \alpha(c_{cr} - c_l) > h$

Since we have

$h_i=(-\lambda_i+\beta_i)c_{pr}+\alpha(c_{cr}-c_i)-g^0$ for $k<i<m$, $h_i$ is constant. Hence, $h_i$ changes its sign at most once in i and $P_{ij}(t)$ is totally positive of order $2(TP_2)$(see[1]). By using the variation diminishing property of $TP_2$ functions, we obtain the result.

## Ⅳ. Discussions

A cost limit replacement policy in redundant system is studied. The system consist of identical redundant units. It is assumed that the state of the system is known only by inspection. When the system is inspected, it is preventively replaced or the system is inspected after time t which depends on the state of the system,. The expected cost rate is used as criterion. we show the some properties of optimal stationary policy and ways to obtain the optimal inspection and replacement limit is proposed. It is important that the control limit rule holds. For further studies, 1) Preventive cost is dependent in the state of system. 2) Operation cost exists.

## Ⅴ. References


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