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# Rank tests for Conparing several treatments with a control in a Randomized Block experiment

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## ABSTRACT

Propose three rank tests based on different kinds of ranking methods for comparing several treatments with a control in a randomized block experiment. Monte Carlo power simulation study is examined in some small sample sizes and configurations to recommend a better test for applications.

## I. Introduction

In order to secure the advantages of increased homogeneity and the resultant increased precision in detecting some of treatment effects, it has been well-known to stratify the populations or to divide the experimental subjects into blocks, using a randomized block design. The present investigation is concerned with three kinds of rank tests of comparing several treatments with a control in randomized block design. As in the literatures (Lehmann (1975)), we can adopt

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three approaches to this problem; the first is the most popular on which is "within blocks' ranking approach used by many authors since Friedman(1937) did. The second, suggested by Hollander(1966) is based on "between blocks" rankings. The third is based on "aligned" ranking scheme considered by Hodges and Lehmann(1962) and Mehra and sarangi(1967) among others.

Many parametric or rank tests based on these three approaches has been proposed in different forms for the omnibus or ordered alternatives. Here we propose three tests in three approaches for comparing several treatments with a control in randomized block experiments. Further this study also investigates the small sample approximate powers of three tests assuming various population models.

## Ⅱ. Test statistics

Let us consider a two factor experiment comprising $n(\geq 2)$ blocks, each block containing$(k+1)(\geq 2)$ plots receiving k different treatments and a control(standard). Like the usual randomized block design model, we express the responses $X_{ij}$ of the plot receiving the j-th treatment in the i-th block as

$$(2.1) \qquad X_{ij}=\mu+\beta_i+\tau_j+\varepsilon_{ij}, \qquad i=1,2,\cdots n;\ j=0.1,\cdots k;$$

where u stands for the mean effect, $\beta_1,\beta_2,\cdots\beta_n$ for the block effects(may or may not be stochastic),$\tau_0,\tau_1,\cdots\tau_k$ for the treatment effects(assumed to be nonstochastic), and $\varepsilon_{ij}$'s are the independent random variables with a common distribution function F and the corresponding density f; $P(X_{ij}\leq x)=F_j(x-\beta_i)$. Without any loss of generality, we may set $\sum_{0\geq j\leq k}\tau_j=0$ and$\sum_{1\geq i\leq n}\beta_i=0$. We consider $B_i$'s as nuisance parameters.

The problem of interest under this setting is to test

$$(2.2) \qquad H_0: \tau_0=\tau_1=\cdots=\tau_k=0$$

against

$$(2.3) \qquad H_1: \tau_0\leq\tau_1,\tau_2,\cdots,\tau_k, \text{ at least one strict inequality.}$$

Under this situation, we consider three kinds of tests based on different ranking schemes.
Remark 1: The alternative hypothesis can be formulated as

$$H_2: \tau_0\geq\tau_1,\tau_2,\cdots,\tau_k, \text{ at least one strict inequality.}$$

### 2.1 "Within-blocks" rank statistics

When we consider the model(2.1), it is meaningless the ranks based on joint ranking because of the block effects. Thus Friedman(1937) first proposed a test based on independent rankings of observations within each block. Since then, many researches have been done for various alternatives. We can adopt this idea and propose a test statistic based on this ranking scheme. Let $R_{ij}(j=0,1,\cdots k)$ be the ranks of $X_{ij}$ among i-th block's observations$(X_{i0},X_{i1},\cdots,X_{ik})$. Then we can form a statistic for detecting differences between treatments and a control in each block as

$$D_i = \sum_{j=1}^{K} (R_{ij} - R_{i0}).$$

Now a test statistic for (2.2) against (2.3) can be

$$T_1 = \sum_{i=1}^{n} D_i. \tag{2.4}$$

The test based on (2.4) rejects $H_0$ for large values of $T_1$. From the well-known asymptotic results of rank statistic, we may use a normal approximation of $T_1$ with

$$E(T_1) = 0, \tag{2.5}$$

$$Var(T_1) = nk(k+1)^2(k+2)/12. \tag{2.6}$$

Remark 2: Similarly Fligner and Wolfe(1982) proposed a test based on "within blocks" ranking with Mann-Whitney statistics for $H_0$ against $H_1$.

### 2.2 "Between blocks" rank statistics

Let $Y_{ij} = |X_{i0} - X_{ij}|$ and $S_{ij}$=rank of $Y_{ij}$ in the ranking from least to greatest of $Y_{ij}; 1=1,2,\cdots,n$. Furthermore, let

$$V_j = \sum_{i=1}^{n} S_{ij}\, \psi_{ij}, \tag{2.7}$$

where $\psi_{ij}=1$ if $X_{i0}<X_{ij}$ and 0 otherwise. Following the works of Hollander (1966), we adopt his lemma 2 here.

Therorem 1: Assume $0< \int F_0\, dF\, j<1$ for $j=1,2,\cdots,k$ and let $Q_j=F_j{*}F_j$,

$W_j = V_j / \binom{n}{2}$  Then $\sqrt{n}\,(W_j - \int Q_0 dQ_j)$ have an asymptotic k-variate normal distribution.

Hollander proposed an asymptotically distribution-free multiple comparison procedure. We propose a test statistics $T_2$ for testing $H_0$ against $H_1$ by modifying his works;

$$(2.8) \qquad T_2 = \sum_{j=1}^{n} W_j.$$

The test based on (2.8) rejects $H_0$ for large values of $T_2$. Hollander showed that the distribution of $V_j (j=1,2,\cdots,k)$ was asymptotically normally distributed with

$$(2.9) \qquad E(V_j) = n(n+1)/4,$$

$$(2.10) \qquad Cov(V_i, V_j) = A_1 + 2A_2 + A_3 - n^2(n+1)^2/16,$$

where $A_1 = n(n-1)/6 + n(n-1)(n-2)\lambda(F) + n(n-1)(n-2)(n-3)/16,$

$A_2 = n(n-1)\mu(F) + n(n-1)(n-2)/8$, $A_3 = n/3 + n(n-1)/4$ and

$\lambda(F) = Pr(X_1 < X_2; X_1 < X_5 + X_6 - X_7),$

$\mu(F) = Pr(X_1 < X_2 + X_3 - X_4; X_1 < X_5 + X_6 - X_7)$

when $X_1, X_2, \cdots, X_7$ are iid according to F. Since the test based on $T_2$ is not distribution-free, we need to estimate $\lambda(F)$ and $\mu(F)$ to obtain an asymptotically distribution-free test. However, estimating these are not an easy job, so we can have an conservative version of $T_2$ by plugging in maximum possible values of $\lambda(F)$ and $\mu(F)$. Lehmann(1964) showed the inequality $6/24 < \lambda(F) \leq 7/24$ and Hollander(1967) showed that $\mu(F) \leq ((\sqrt{2} + 6)/24) = .3089$. Thus we can have a conservative test based on $T_2^*$ by setting $\lambda(F) = 7/24$ and $U(F) = 0.3089$.

### 2.3. "aligned" rank statistics

Hodge and lehmann(1962) suggested a conditional ranking procedure after alignment. This procedure seems to be combinations of the above two procedures; the first step is to bring the observations in the various blocks into alignment with one another. This is done by subtracting from each observation the mean observation in its block, but some cases other methods for alignment might be better, such as subtracting median or trimmed mean. Once the observations are aligned, then they are pooled and ranked without regard to their blocks. Let $\hat{R}_{ij}$ be the rank of $X_{ij}$ ofter alignment and $\hat{R}_j$ be the rank sum of $\hat{R}_{ij}$ from $\hat{R}_{1j},\hat{R}_{2j},\cdots,\hat{R}_{nj}$, then we propose the test statistic for $H_0$ against $H_1$ as followed;

$$T_3=\sum_{j=1}^{k}(\hat{R}_j-\hat{R}_{\cdot 0}). \tag{2.13}$$

The test based on $T_3$ rejects $H_0$for large values of $T_3$. The test based on $T_3$ is the conditional test, which modifies the test proposed by Hodge and Lehmann(1962) for the omnibus alternatives to the one-sided alternatives(2.3). From Mehra and Sarange(1967) we may have the following theorem.

Theorem 2: Let $E_c(\cdot)$, $Var_c(.,.)$ stand for the conditional expectation, variance and covariance respectively, under $H_0$, given a configuration. Then for $j=0,1,\cdots,k$,

$$E_c(\hat{R}_{\cdot j})=\sum_{i=1}^{n}\bar{R}_i \tag{2.14}$$

$$Var_c(\hat{R}_{\cdot j})=\sum_{i=1}^{n}\sigma_{ci}^2; \tag{2.15}$$

$$Cov_c(\hat{R}_{\cdot j}\hat{R}_{\cdot j'})=\sum_{i=1}^{n}[\sigma_{ci}^2/k]; \tag{2.16}$$

where $\bar{R}_i=\sum_{j=0}^{k}R_{ij}/(k+1),\sigma_{ci}^2=\sum_{j=0}^{k}(\hat{R}_{ij}-\bar{R}_i)^2/(k+1)$.

From theorem 2, we can easily have $E_c(T_3)=0$, $Var_c(T_3)=(K^2+3)\sum_{i=1}^{n}\sigma_{ci}^2$.

Let $T_m$ stands for the vectors $(T_{n0},T_{n1},\cdots,T_{nk})'$ respectively, where

$$T_{nj}=\frac{\sqrt{k}\{R_j-n(n(k+1)+1).2\}}{\{(k+1)(\sum_{i=1}^{n}\delta^2_{Ci})\}^{1/2}} \tag{2.17}$$

and $\Delta$ stands for the covariance matrix $\| \delta_{jj'}-1/(k+1) \|$, where $\delta_{jj'}$ is the kronecker's delta. By using some discussion of Mehra and Sarangi on p.97, we can have the following theorem easily.

Theorem 3: Under $H_0$ the random vector $T_n$ converges in distribution, as $n\to\infty$, to a multivariate normal vector $N(0, \Delta)$.

Following the asymptotic results of theorem 3, we may have the asymptotic normality of test statistics $T_3$ given a configuration as $n \to \infty$ and complete a testing procedures of $H_0$ against $H_1$.

## Ⅲ. Simulation

A simulation study is undertaken to compare the approximate power performance of the tests considered here. We assume that one observation is in each cell and sizes of biock are 10 and 15.

The distributions, used to generate the samples, are uniform, normal, Cauchy and exponential. The scale parameters for all these distributions are given as 1. SAS IML is used to generate the samples.

The study includes 3 values for the number of treatments, namely 3,4 and 5. The approximate 0.05 and 0.01 critical values of discussed tests are used. The sizes and powers of each test are estimated using 1000 repetitions.

Three types of $\tau_i$'s configurations under $H_1$ are used as

Type 1 : $\tau_0 < \tau_1 = \cdots = \tau_k$;

$\tau_0 = -0.k$, $\tau_i = 0.1$;

Type 2 : $\tau_0 < \tau_1 < \cdots < \tau_k$;

$\tau_0 = -0.1$, $\tau_i = 0.0$, $\tau_k = 0.1$

Type 3 : $\tau_0 = \tau_{\cdot} \cdots < \tau_k$;

$\tau_0 = -0.1$, $\tau_i = -0.1$, $\tau_k = 0.k$

$B_i$'s configurations are given by

$n=10, (\beta_1 \beta_5) = (-5 : -1(1)); (\beta_6 : \beta_{10}) = (1:5(1))$,

$n=15$, $(\beta_1 : \beta_7) = (-7 : -1(1)); \beta_8 = 0; (\beta_9 : \beta_{15}) = (1:7(1))$. Further let $u=0$.

The values in the parenthesis are the approximate powers at $\alpha = 0.01$.

(i) Approximate power estimates for the case of $k=3$, $n=10$

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.05(0.003) | 0.04(0.004) | 0.04(0.006) |
| Type 1 | 0.27(0.06) | 0.244(0.07) | 0.3(0.08) |
| Type 2 | 0.168(0.035) | 0.163(0.046) | 0.17(0.047) |
| Type 3 | 0.1(0.12) | 0.087(0.018) | 0.092(0.023) |
| Uniform | | | |
| $H_0$ | 0.074(0.014) | 0.06(0.018) | 0.064(0.016) |
| Type 1 | 0.93(0.67) | 0.95(0.84) | 0.96(0.85) |
| Type 2 | 0.65(0.27) | 0.71(0.33) | 0.73(0.40) |
| Type 3 | 0.23(0.05) | 0.17(0.038) | 0.22(0.044) |
| Cauchy | 0.064(0.008) | 0.064(0.006) | 0.056(0.014) |
| $H_0$ | | | |
| Type 1 | 0.16(0.034) | 0.12(0.034) | 0.13(0.04) |
| Type 2 | 0.10(0.008) | 0.068(0.008) | 0.075(0.01) |
| Type 3 | 0.122(0.012) | 0.08(0.014) | 0.08(0.024) |
| Exponential | 0.04(0.006) | 0.05(0.007) | 0.05(0.01) |
| $H_0$ | | | |
| Type 1 | 0.43(0.16) | 0.37(0.15) | 0.39(0.14) |
| Type 2 | 0.28(0.074) | 0.22(0.063) | 0.24(0.065) |
| Type 3 | 0.15(0.023) | 0.1(0.03) | 0.12(0.026) |

(ii) Approximate power estimates for the case of $k=4$, $n=10$

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.045(0.005) | 0.04(0.007) | 0.042(0.008) |
| Type 1 | 0.346(0.132) | 0.364(0.125) | 0.396(0.173) |
| Type 2 | 0.16(0.04) | 0.18(0.04) | 0.195(0.05) |
| Type 3 | 0.094(0.016) | 0.09(0.016) | 0.092(0.016) |
| Uniform | | | |
| $H_0$ | 0.038(0.01) | 0.038(0.006) | 0.038(0.004) |
| Type 1 | 0.988(0.93) | 0.996(0.984) | 0.998(0.982) |
| Type 2 | 0.636(0.33) | 0.73(0.336) | 0.74(0.392) |

| | | | |
|---|---|---|---|
| Type 3 | 0.174(0.04) | 0.14(0.028) | 0.2(0.044) |
| Cauchy | | | |
| $H_0$ | 0.048(0.01) | 0.054(0.014) | 0.062(0.016) |
| Type 1 | 0.194(0.07) | 0.16(0.048) | 0.142(0.056) |
| Type 2 | 0.068(0.016) | 0.064(0.01) | 0.07(0.018) |
| Type 3 | 0.072(0.012) | 0.054(0.008) | 0.064(0.016) |
| Exponential | | | |
| $H_0$ | 0.042(0.009) | 0.035(0.006) | 0.037(0.004) |
| Type 1 | 0.56(0.326) | 0.5(0.256) | 0.55(0.3) |
| Type 2 | 0.22(0.08) | 0.2(0.07) | 0.213(0.07) |
| Type 3 | 0.114(0.02) | 0.108(0.017) | 0.12(0.017) |

(iii) Approximate power estimates for the case of $k=5$, $n=10$

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.066(0.01) | 0.059(0.012) | 0.06(0.014) |
| Type 1 | 0.5(0.2) | 0.5(0.21) | 0.515(0.23) |
| Type 2 | 0.26(0.08) | 0.266(0.08) | 0.28(0.085) |
| Type 3 | 0.11(0.022) | 0.105(0.02) | 0.11(0.027) |
| Uniform | | | |
| $H_0$ | 0.04(0.008) | 0.045(0.009) | 0.052(0.012) |
| Type 1 | 1.0(0.996) | 1.0(1.0) | 1.0(1.0) |
| Type 2 | 0.94(0.69) | 0.944(0.65) | 0.95(0.76) |
| Type 3 | 0.196(0.046) | 0.118(0.03) | 0.156(0.042) |
| Cauchy | | | |
| $H_0$ | 0.048(0.01) | 0.04(0.008) | 0.058(0.01) |
| Type 1 | 0.23(0.06) | 0.18(0.038) | 0.148(0.05) |
| Type 2 | 0.13(0.038) | 0.106(0.030) | 0.11(0.022) |
| Type 3 | 0.076(0.01) | 0.054(0.008) | 0.06(0.02) |
| Exponential | | | |
| $H_0$ | 0.058(0.009) | 0.056(0.007) | 0.052(0.005) |
| Type 1 | 0.69(0.42) | 0.59(0.33) | 0.65(0.4) |
| Type 2 | 0.42(0.18) | 0.34(0.13) | 0.38(0.14) |
| Type 3 | 0.117(0.028) | 0.087(0.02) | 0.09(0.02) |

(iv) Approximate power estimates for the case of k=3, n=15

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.053(0.006) | 0.04(0.005) | 0.04(0.004) |
| Type 1 | 0.315(0.11) | 0.36(0.115) | 0.36(0.13) |
| Type 2 | 0.215(0.056) | 0.227(0.058) | 0.233(0.064) |
| Type 3 | 0.104(0.022) | 0.093(0.017) | 0.103(0.02) |
| Uniform | | | |
| $H_0$ | 0.054(0.02) | 0.052(0.012) | 0.056(0.016) |
| Type 1 | 0.978(0.898) | 0.998(0.96) | 0.998(0.96) |
| Type 2 | 0.79(0.504) | 0.87(0.58) | 0.87(0.612) |
| Type 3 | 0.286(0.108) | 0.276(0.078) | 0.324(0.106) |
| Cauchy | | | |
| $H_0$ | 0.062(0.012) | 0.054(0.014) | 0.058(0.014) |
| Type 1 | 0.14(0.026) | 0.12(0.016) | 0.11(0.02) |
| Type 2 | 0.12(0.028) | 0.11(0.018) | 0.1(0.02) |
| Type 3 | 0.064(0.01) | 0.054(0.01) | 0.064(0.014) |
| Exponential | | | |
| $H_0$ | 0.05(0.013) | 0.05(0.011) | 0.053(0.01) |
| Type 1 | 0.535(0.277) | 0.46(0.21) | 0.5(0.23) |
| Type 2 | 0.336(0.122) | 0.274(0.087) | 0.3(0.1) |
| Type 3 | 0.142(0.024) | 0.114(0.021) | 0.12(0.002) |

(v) Approximate power estimates for the case of k=4, n=15

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.04(0.007) | 0.05(0.008) | 0.055(0.01) |
| Type 1 | 0.08(0.009) | 0.09(0.01) | 0.11(0.026) |
| Type 2 | 0.07(0.01) | 0.09(0.015) | 0.09(0.017) |
| Type 3 | 0.075(0.012) | 0.105(0.018) | 0.101(0.026) |
| Uniform | | | |
| $H_0$ | 0.046(0.014) | 0.05(0.008) | 0.044(0.008) |
| Type 1 | 1.0(1.0) | 1.0(1.0) | 1.0(1.0) |

| | | | |
|---|---|---|---|
| Type 2 | 0.76(0.54) | 0.86(0.6) | 0.86(0.63) |
| Type 3 | 0.24(0.076) | 0.23(0.044) | 0.3(0.074) |
| Cauchy | | | |
| $H_0$ | 0.042(0.01) | 0.054(0.012) | 0.05(0.01) |
| Type 1 | 0.19(0.08) | 0.16(0.04) | 0.14(0.044) |
| Type 2 | 0.11(0.04) | 0.11(0.026) | 0.1(0.026) |
| Type 3 | 0.06(0.02) | 0.06(0.01) | 0.064(0.012) |
| Exponential | | | |
| $H_0$ | 0.043(0.005) | 0.058(0.009) | 0.053(0.009) |
| Type 1 | 0.23(0.066) | 0.237(0.062) | 0.23(0.068) |
| Type 2 | 0.1(0.017) | 0.118(0.017) | 0.1(0.016) |
| Type 3 | 0.115(0.02) | 0.13(0.025) | 0.116(0.02) |

(vi) Approximate power estimates for the case of $k=5$, $n=15$

| Distributions | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| Normal | | | |
| $H_0$ | 0.052(0.01) | 0.048(0.006) | 0.054(0.01) |
| Type 1 | 0.61(0.343) | 0.656(0.35) | 0.66(0.37) |
| Type 2 | 0.295(0.118) | 0.343(0.1) | 0.344(0.11) |
| Type 3 | 0.094(0.027) | 0.1(0.019) | 0.102(0.026) |
| Uniform | | | |
| $H_0$ | 0.054(0.008) | 0.052(0.006) | 0.048(0.006) |
| Type 1 | 1.0(1.0) | 1.0(1.0) | 1.0(1.0) |
| Type 2 | 0.98(0.87) | 0.988(0.886) | 0.992(0.914) |
| Type 3 | 0.21(0.07) | 0.17(0.038) | 0.22(0.06) |
| Cauchy | | | |
| $H_0$ | 0.058(0.01) | 0.05(0.008) | 0.056(0.014) |
| Type 1 | 0.31(0.13) | 0.24(0.07) | 0.2(0.05) |
| Type 2 | 0.15(0.04) | 0.14(0.03) | 0.12(0.03) |
| Type 3 | 0.098(0.01) | 0.08(0.012) | 0.084(0.022) |
| Exponential | | | |
| $H_0$ | 0.57(0.013) | 0.048(0.008) | 0.045(0.011) |
| Type 1 | 0.817(0.64) | 0.769(0.51) | 0.822(0.59) |
| Type 2 | 0.5(0.291) | 0.435(0.194) | 0.47(0.234) |
| Type 3 | 0.129(0.043) | 0.117(0.029) | 0.128(0.034) |

# Ⅳ. Discussion & Conclusion

we considered three rank tests based on different ranking methods for comparing several treatments with a control in randomized block experiments. Three methods themselves have own benefits and shortcomings; that is,

1) the test based on "within blocks ranking" is easy to use and good powers in the case of Cauchy of exponential distributions, but it shows lower powers comparing with the others when the probability models are normal and uniform.
2) The test based on "between blocks ranking" utilizes more information and has a reasonable power(most cases lie between $T_1$ and $T_3$) under three types of $H_1$ configurations, but it is very complex to evaluate test statistic,
3) the test based on "aligned ranking" looks good when the probability models are normal and uniform but it also needs two-step calculations.

Unfortunately we could not find the test which dominates other tests. However, we can suggest the followings in view of this simulation results; the test based on "within blocks" ranking when the probability models are exponential, Cauchy, lognormal and weibull(did partially). Further if the probability models are normal, uniform, contaminated normal(did partially), we found the test based on "aligned" ranking shows superior to other tests. But these two approximate power differences are not much, so we would recommend to use the test based on T ("within blocks" rank statistic) if experimenters do not have any idea about the probability models.

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