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TRANSNORMAL SYSTEMS ON KP^2

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1. Introduction

Some of geometric properties of submanifolds can be determined by their topological properties of ambient spaces. Isoparametric hypersurfaces in spheres are good examples. They are closely related to transnormal systems which are introduced by J. Bolton. In fact, transnormal systems give singular foliations with some restrictions on their singular foils. And transnormal systems on complex and quaternionic projective spaces were studied by K. Park. The method used in [5] is based on the observation that those projective spaces are images of Riemannian submersions of spheres. But the Cayley projective plane KP^2 is not the image of a Riemannian submersion of sphere. And this is the only compact, simply connected symmetric space of rank one which is not included in [5].

In this paper, we investigate homology groups of hypersurfaces in trans-normal systems on KP^2. Also we check restrictions on singular foils. In particular, we concentrated on the restrictions on codimensions of singular foils. Actually there are a few possible cases of transnormal systems on KP^2. In fact, this is a revised version of [6]. Also we will give examples of these possible cases.

2. Preliminaries

A *transnormal system* on a complete connected Riemannain manifold N is a smooth partition of N into connected submanifolds, called *foils*, such that any geodesic of N meets each foil orthogonally at none or all of its points.

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We need the following propositions.

Proposition 2.1. ([1]) *Let $\mathcal{L}$ be a transnormal system on N with a foil M of codimension 1. Then one of the following holds*

(1) *If it has no singular foils, then it is a codimension-one foliation,*

(2) *If it has only one singular foil, then N has a vector bundle structure over the singular foil or N has a double cover, and*

(3) *If it has only two singular foils, say. A_1 and A_2, then N has the bundle decomposition $DA_1 \bigcup_M DA_2$ of two sphere bundles DA_i over A_i, glued together along their common boundary M.*

Proposition 2.2. ([8]) *A closed manifold N has a codimension-one foliation if and only if its Euler characteristic is zero.*

3. Transnormal systems on KP^2

Let $\mathcal{L}$ be a transnormal system on KP^2 with a codimension-one foil M. Note that the integral homology group of KP^2 is given as follows :

$$H_i(KP^2) = \begin{cases} Z & \text{for } i = 0, 8, 16 \\ 0 & \text{otherwise} \end{cases}$$

Thus the case (1) of Proposition 2.1 can not happen. Since KP^2 is compact, it can not have a vector bundle structure. By using Gysin exact sequence with Z_2 coefficients, it also can not have a double cover. Hence we have the following

Proposition 3.1. *Let $\mathcal{L}$ be a transnormal system on KP^2. Then it has exactly two singular foils.*

Now we assume that $\mathcal{L}$ is a transnormal system on KP^2 with a hypersurface M and two singular foils A_1 and A_2. Let k_i be the codimension of A_i, $(i = 1, 2)$. Then we have the bundle decomposition $KP^2 = DA_1 \bigcup_M DA_2$.

Let $\phi_i : M \longrightarrow A_i$ be the canonical map. Over each path-component of each A_i the homotopy fiber of ϕ_i has weak homotopy type of a sphere. So KP^2 generalizes to the double mapping cylinder $DM = A_1 \bigcup_{\phi_1} (M \times I) \bigcup_{\phi_2} A_2$ and $j : M \hookrightarrow KP^2$ becomes the inclusion $x \longmapsto (x, 1/2)$ of M into DM. Let F be the path-component of the homotopy fiber of j. Then $H_*(F; Z)$ is given as follows (cf. [4]) :

(k_1, k_2)	$H_i(F;Z)$	
$k_1 \neq k_2$	Z	$i = 0$ or $i \equiv k_1, k_2 \bmod (k_1 + k_2)$
no twists	$Z \oplus Z$	$i > 0$ and $i \equiv 0 \bmod (k_1 + k_2)$
$k_1 = k_2$	Z	$i = 0$
no twists	$Z \oplus Z$	$i > 0$ and $i \equiv 0 \bmod k_1$
$k_1 > k_2 = 1$	Z	$i = 0$ or $i \equiv \pm 1 \bmod (2k_1 + 2)$
ϕ_1 twisted	$Z \oplus Z$	$i > 0$ and $i \equiv 0 \bmod (2k_1 + 2)$
ϕ_2 not twisted	Z_2	$i \equiv k_1, k_1 + 1 \bmod (2k_1 + 2)$
$k_1 = k_2 = 1$	Z	$i = 0$ or $i \equiv 3 \bmod 4$
ϕ_1 twisted	$Z \oplus Z_2$	$i \equiv 1 \bmod 4$
ϕ_2 not twisted	Z_2	$i \equiv 2 \bmod 4$
	$Z \oplus Z$	$i > 0$ and $i \equiv 0 \bmod 4$
$k_1 = k_2 = 1$	Z	$i = 0$
ϕ_1, ϕ_2	$Z \oplus Z$	$i > 0$ and $i \equiv 0 \bmod 3$
both twisted	$Z_2 \oplus Z_2$	$i \equiv 1 \bmod 3$

Table 1. $H_*(F;Z)$

Since $H_*(F) \otimes H_*(KP^2)$ is a term of a spectral sequence which converges to $H_*(M)$. Thus we can compute the homology of M by Table 1. Note that $\dim M = 15$ and hence $H_i(M) = H_{15-i}(M)$ by the Poincare duality. Using this fact, we can exclude almost all of the possibilities of (k_1, k_2). The computation is trivial. One of the possible cases is the case $k_1 = k_2 = 1$ with both ϕ_i twisted. But, in this case, we have $H_i(M) = Z$ for every $1 \leq i \leq 15$ which is impossible topologically. Hence we have

Proposition 3.2. *Let $\mathcal{L}$ be a transnormal system on KP^2 with a hypersurface. Then its one of two singular foils has codimension 7 and the other one has 15 or 4.*

If A_1 has codimension 7, then it is an 8-sphere. And if A_2 has codimension 15, then it is a point. This case exists (cf. [8]). On the other hand, if A_2 has codimension 4, then it is an 11-sphere. But we still don't know about the existence of this case.

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