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ON SPECIAL PROJECTIVE KILLING 2-FORM IN SASAKIAN MANIFOLDS

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1. Introduction

Let M be an n-dimensional Riemannian manifold. Many authors have studied some kinds of vector fields which have geometric significances such as Killing, conformal Killing and projective Killing vector fields ([1], [4], [7], [8]).

Also, the vector fields were generalized to the differential forms of degree p ($p \geq 1$) respectively in M. Making use of them, we have obtained some conditions for a complete simply connected Riemannian manifold to be isometric with a sphere and many other properties.

On the other hand, the following theorems play very important roles in the proof that M is isometric with a sphere.

Theorem A ([5], [10]). *Let M be a complete connected Riemannian manifold of dimension $n(n \geq 2)$. In order for M to admit a non-trivial solution ϕ for the system of differential equations*

$$\nabla_a \nabla_b \phi + k\phi g_{ab} = 0, \tag{1.1}$$

it is necessary and sufficient that M be isometric with a sphere S^n of radius $1/\sqrt{k}$ in E^{n+1}, where k is positive constant.

The function ϕ satisfying (1.1) is called a special concircular scalar field.

Theorem B([5], [9]). *Let M be a complete simply connected Riemannian manifold of dimension n. In other for M to admit a non-trivial solution ϕ for the system of different equations*

$$\nabla_a \nabla_b \nabla_c \phi + k(2g_{bc}\nabla_a \phi + g_{ac}\nabla_b \phi + g_{ab}\nabla_c \phi) = 0, \tag{1.2}$$

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it is necessary and sufficient that M be isometric with a sphere S^n of radius $1/\sqrt{k}$ in E^{n+1}, where k is positive constant.

Recently, J. B. Jun and S. Yamaguchi has introduced the notion of a special projective Killing p-form ($p \geq 2$) and found some kinds of geometric meaning [2].

The purpose of this paper is to find the non-trivial function corresponding to (1.1) for a special projective Killing 2-form that M is isometric with a sphere.

That is, we will prove the following:

Theorem. *Let M be a complete connected Sasakian manifold of dimension n admitting a special projective Killing 2-form $d\theta$ with 1. If the scalar field $\Lambda d\theta + 6i(\eta)\theta$ is not constant, then M is isometric with a unit sphere in E^{n+1}.*

We recall some results concerning to the conformal Killing, special Killing and projective Killing p-forms in section 3. The proof of theorem is given in section 4.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold. Taking its orientable double covering if necessary, we may consider M is orientable without loss of generality. Throughout this paper, we assume that manifolds are connected and class of C^∞. Denote respectively by $g_{ab}, R_{abc}{}^e$, the metric, the curvature of M in terms of local coordinates $\{x^a\}$, where Latin indices run over the range $\{1, 2, \cdots, n\}$.

We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential p-form

$$u = (1/p!)u_{a_1 \cdots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}$$

with skew symmetric coefficients $u_{a_1 \cdots a_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given by

$$\begin{aligned} (du)_{a_1 \cdots a_{p+1}} &= \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{a_i} u_{a_1 \cdots \hat{a}_i \cdots a_{p+1}}, \\ (\delta u)_{a_2 \cdots a_p} &= -\nabla^r u_{r a_2 \cdots a_p}, \end{aligned}$$

where $\nabla^r = g^{rs}\nabla_s, \nabla_s$ denotes the operator of covariant differentiation and $\hat{a}_i$ means a_i to be deleted.

An n-dimensional Riemannian manifold M is called a Sasakian manifold if there exists a unit special Killing 1-form η with constant 1, that is

$$\nabla_a \nabla_b \eta_c = \eta_b g_{ac} - \eta_c g_{ab}.$$

Then n is necessarily odd and M is orientable. With respect to a local coordinate system $\{x^a\}$, if we define a 2-form $\phi = \frac{1}{2}\phi_{ab} dx^a \wedge dx^b$ by $\phi_{ab} = \nabla_a \eta_b$, then we have $d\eta = 2\phi$ and it holds

$$\nabla_a \phi_{bc} = \eta_b g_{ac} - \eta_c g_{ab}. \tag{2.1}$$

We denote by $i(\eta)$ and Λ the inner product of 1-form η and 2-form $d\eta(= 2\phi)$. Then, for any p-form u the operators $i(\eta)$ and Λ are defined by

$$\begin{aligned} (i(\eta)u)_{a_2 \cdots a_p} &= \eta^r u_{ra_2 \cdots a_p} & (p \geq 1), \\ i(\eta)u &= 0 & (p = 0), \\ (\Lambda u)_{a_3 \cdots a_p} &= \phi^{rs} u_{rsa_3 \cdots a_p} & (p \geq 2), \\ \Lambda u &= 0 & (p = 0, 1). \end{aligned}$$

We call u as a conformal Killing 2-form [6] if there exists a 1-form θ_a such that

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2\theta_c g_{ab} - \theta_a g_{bc} - \theta_b g_{ac}.$$

This θ_a is called the associated 1-form of u_{ab}. If θ_a vanishes identically, then u is called a Killing 2-form [1].

In a Sasakian manifold, the following theorem is well known.

Theorem C[11]. *Let M be a complete simply connected Sasakian space $(n > 3)$ admitting a conformal Killing tensor u_{ab} whose associated vector is θ_a. If $< \theta, \theta >$ or $< \theta, \eta >$ is not constant, then M is isometric with a unit sphere.*

If a Killing 2-form u satisfies

$$\nabla_c \nabla_b u_{ad} + k(g_{cb} u_{ad} - g_{ca} u_{bd} - g_{cd} u_{ab}) = 0,$$

where k is constant, then it is called a special Killing 2-form with constant k [8].

As for a special Killing 2-form, the following theorem was known.

Theorem D [8]. *Let M be a complete simply connected Riemannian manifold admitting special Killing 2-forms u and v with a positive constant* *k. If their inner product is not constant, then M is isometric with a sphere of radius* $1/\sqrt{k}$ *in* E^{n+1}.

Remark. The above theorem also was proved for the general degree $p(p \geq 1)$.

Moreover, we call u as a projective Killing 2-form [7], if there exists a 1-form θ_a called the associated 1-form such that

$$\begin{aligned}\nabla_c\nabla_b u_{ad} - R_{bac}{}^e u_{ed} - \frac{1}{2}(R_{bcd}{}^e u_{ae} + R_{cad}{}^e u_{be} + R_{bad}{}^e u_{ce}) \\ = (g_{ba}\nabla_c\theta_d + g_{ca}\nabla_b\theta_d - g_{bd}\nabla_c\theta_a - g_{cd}\nabla_b\theta_a).\end{aligned}$$

Remark. In the above definition, a 2-form u is said to be projective Killing of first(resp. second) kind if $\delta u + n\theta$ does not vanish (resp. vanishes) identically.

Especially, for any projective Killing 2-form of first kind and second kind we have the followings respectively [3].

Theorem E. *Let M be an* $n(n > 3)$*-dimensional complete connected Sasakian manifold. If it admits a non-Killing projective Killing 2-form of first kind, then M is isometric with a unit sphere.*

Theorem F. *An* $n(n > 5)$*-dimensional complete simply connected Sasakian manifold M is isometric with a unit sphere* S^n *if M admits projective Killing 2-form u of second kind and the function* $|d\theta|^2 + 18|\theta|^2$ *is not constant for the associated 1-form* θ *of u.*

We call an exact 2-form $d\theta$ as special projective Killing with constant k[2], if it satisfies

$$\begin{aligned}\nabla_a\nabla_b(d\theta)_{cd} + k(g_{ab}(d\theta)_{cd} + g_{ac}(d\theta)_{db} + g_{ad}(d\theta)_{bc}) \\ 3k(g_{ac}\nabla_b\theta_d - g_{bd}\nabla_a\theta_c + g_{bc}\nabla_a\theta_d - g_{ad}\nabla_b\theta_c = 0,\end{aligned} \tag{2.2}$$

$$\nabla_b(d\theta)_{cd} + \nabla_c(d\theta)_{bd} - 3(\nabla_b\nabla_c\theta_d + k\theta_b g_{cd}) = 0. \tag{2.3}$$

For a special projective Killing 2-form, we obtained the following [2].

Theorem G. *Let M be a complete connected Riemannian manifold of dimension n admitting a special projective Killing 2-form* $d\theta$ *with positive constant k. If* $\delta\theta$ *is not constant, then M is isometric with a sphere* S^n *of radius* $1/\sqrt{k}$ *in* E^{n+1}.

Furthermore, we have found a non-trivial function corresponding to ϕ for the system of differential equations (1.2) as

Theorem H. *Let M be a complete connected Riemannian manifold of dimension n adimitting a special projective Killing p-form $d\theta(2 \le p < n)$ with positive constant k. If $|d\theta|^2 + p(p+1)^2 k|\theta|^2$ is not constant, then M is isometric with a sphere S^n of radius $1/\sqrt{k}$ in E^{n+1}.*

3. Proof of Theorem

In this section, we devote ourselves to give the geometric meaning with respect to special projective Killing 2-form $d\theta$ in an n-dimensional Sasakian manifold. In othere words, it might be interesting to find another non-trivial function corresponding to ϕ for the system of differential equations (1.1) that a Sasakian manifold admitting a special projective Killing 2-form with constant k is isometric with a sphere.

We put $\alpha = i(\eta)\theta = \eta^r\theta_r$. Then it can readily be verified from (2.1) that

$$\nabla_b\nabla_c\alpha = (\nabla_b\nabla_c\theta_r)\eta^r + (\nabla_c\theta_r)\phi_b^r + (\nabla_b\theta_r)\phi_c^r + \eta_c\theta_b - \alpha g_{bc}.$$

Hence we have,

$$(3.1) \qquad \begin{aligned}&(\nabla_c\theta_r)\phi_b^r + (\nabla_b\theta_r)\phi_c^r \\ &\quad = \nabla_b\nabla_c\alpha - (\nabla_b\nabla_c\theta_r)\eta^r + \alpha g_{bc} - \eta_c\theta_b.\end{aligned}$$

Next we put $\beta = \Lambda d\theta = \phi^{rs}(d\theta)_{rs}$. By making use of (2.1), it is clear that

$$\nabla_c\beta = \nabla_c(d\theta)_{rs}\phi^{rs} + 2(d\theta)_{rc}\eta^r.$$

We differentiate the above equation covariantly and take account of (2.1) and (2.2). Then we have

$$\begin{aligned}\nabla_b\nabla_c\beta &= -k(g_{bc}\beta + 2\phi_b^r(d\theta)_{rc}) - 6k((\nabla_c\theta_r)\phi_b^r + (\nabla_b\theta_r)\phi_c^r) \\ &\quad -2(\nabla_b(d\theta)_{cr}\eta^r + \nabla_c(d\theta)_{br}\eta^r) + 2(d\theta)_{rc}\phi_b^r,\end{aligned}$$

which together with (2.3) and (3.1) implies

$$(3.2) \qquad \begin{aligned}\nabla_b\nabla_c\beta &= 2(k-1)(3(\nabla_b\nabla_c\theta_r)\eta^r - \phi_b^r(d\theta)_{rc}) \\ &\quad -k(\beta + 6\alpha)g_{bc} - 6k\nabla_b\nabla_c\alpha.\end{aligned}$$

Hence, we have for the case of $k = 1$ at (3.2)

$$\nabla_b\nabla_c(\beta + 6\alpha) + (\beta + 6\alpha)g_{bc} = 0.$$

which implies that the scalar field $\beta + 6\alpha$ is special concircular. Thus $\beta + 6\alpha = \Lambda d\theta + 6i(\eta)\theta$ is satisfied with (1.1). This completes the proof.

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