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L^2-TRANSVERSE HARMONIC FIELDS ON COMPLETE FOLIATED RIEMANNIAN MANIFOLDS*

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We discuss transverse harmonic vector fields with finite global norms on complete foliated Riemannian manifolds. Our main method is the cut-off function trick.

0. On a compact foliated Riemannian manifolds, geometric transverse fields, that is, transverse Killing, affine, projective, conformal fields have been studied by Kamber and Tondeur([4]), Molino([8]), Pak and Yorozu([10]), Park and Yorozu([12]) and others. In the case of foliations by points, transverse fields are usual fields on Riemannian manifolds. In [11] we considered the transverse harmonic fields on compact foliated Riemannian manifolds and obtained natural extension to well-known results for harmonic fields on Riemannian manifolds. Our main purpose is to study transverse harmonic fields on complete (non-compact) foliated Riemannian manifolds. To do this, we have to mention the notion of "*L^2-transverse fields*" that is, transverse fields with finite global norms. L^2-transverse Killing and conformal fields are already dealt in [1] and [21]. In this paper, we discuss L^2-transverse harmonic fields on complete foliated Riemannian manifolds such that the foliation is minimal and the metric is bundle-like with respect to the foliation, and then the following theorems are proved:

Theorem A. *Let (M, g_M, F) be a Riemannian manifold with a minimal foliation F and a complete bundle-like metric g_M with respect to F. Let $s \in \overline{V}(F)$ be an L^2-transverse field of F. Then s is a transverse harmonic field of F if and only if $\Delta_D(s) + \rho_D(s) = 0$, where $\rho_D(s)$ is the transverse Ricci operator of F and $\Delta_D(s)$ is the Laplacian acting on $\Omega^r(M, Q)$.*

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Theorem B. *Let (M, g_M, F) be as Theorem A. If the transverse Ricci operator ρ_D is non-negative every where in M, then every L^2-transverse harmonic field is D-parallel. If ρ_D is non-negative everywhere and positive for at least one point of M, then any L^2-transverse harmonic field other than zero does not exist in M.*

We shall be in C^∞-category and deal only with connected and oriented manifolds without boundary. We use the following convention on the range of indices:

$$1 \leq i, j \leq p; p+1 \leq a, b, c, d \leq p+q.$$

The Einstein summation convention will be used with respect to those systems of indices.

1. Let (M, g_M, F) be a $(p+q)$-dimensional Riemannian manifold with a foliation F of codimension q and a complete bundle-like metric g_M with respect to F([14]). We assume that F is an oriented foliation([15]). Let ∇ be the Levi-Civita connection with respect to g_M. Then the tangent bundle TM over M has an integrable subbundle E which is given by F. The normal bundle Q of F is defined by $Q = TM/E$. We have a splitting σ of the exact sequence

$$0 \longrightarrow E \longrightarrow TM \underset{\overset{\longleftarrow}{\sigma}}{\overset{\pi}{\longrightarrow}} Q \longrightarrow 0$$

where $\sigma(Q)$ is the orthogonal complement bundle $E^\perp$ of E in TM([3]). Then g_M induces a metric g_Q on Q:

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)), \quad s, t \in \Gamma(Q), \tag{1.1}$$

where $\Gamma(*)$ denotes the set of all sections of $*$. In a flat chart $U(x^i, x^a)$ with respect to F([14]), a local frame $\{X_i, X_a\} = \{\partial/\partial x^i, \partial/\partial x^a - A_a^j \partial/\partial x^j\}$ is called the *basic adapted frame* to F ([8], [13], [16]). Here A_a^j are functions on U with $g_M(X_i, X_a) = 0$. It is clear that $\{X_i\}$ (resp. $\{X_a\}$) spans $\Gamma(E|_U)$ (resp. $\Gamma(E^\perp|_U)$). We omit "$|_U$" for simplicity. We set

$$\begin{aligned} g_{ij} &= g_M(X_i, X_j), \quad g_{ab} = g_M(X_a, X_b) \\ (g^{ij}) &= (g_{ij})^{-1}, \qquad (g^{ab}) = (g_{ab})^{-1} \end{aligned} \tag{1.2}$$

A connection D in Q is defined by

$$\begin{aligned} D_X s &= \pi([X, Y]), X \in \Gamma(E), \quad s \in \Gamma(Q) \text{ with } \pi(Y) = s \\ D_X s &= \pi(\nabla_X Y_s), X \in \Gamma(E^\perp), \quad s \in \Gamma(Q) \text{ with } Y_s = \sigma(s) \end{aligned} \tag{1.3}$$

([3]). Then the connection D in Q is torsion-free and metrical with respect to g_Q ([3]). The curvature R_D of D is defined by

$$(1.4) \qquad R_D(X,Y)s = D_X D_Y s - D_Y D_X s - D_{[X,Y]}s$$

for any $X, Y \in \Gamma(TM)$ and $s \in \Gamma(Q)$. Since $i(X)R_D = 0$ for any $X \in \Gamma(E)$([3]), we can define the Ricci operator $\rho_D : \Gamma(Q) \to \Gamma(Q)$ of E by

$$(1.5) \qquad \rho_D(s) = g^{ab} R_D(\sigma(s), \pi(X_a))\pi(X_b)$$

([4]).

Let $V(F)$ be the space of all vector fields Y on M satisfying

$$(1.6) \qquad [Y, Z] \in \Gamma(E)$$

for any $Z \in \Gamma(E)$. An element of $V(F)$ is called an *infinitesimal automorphism* of F ([4],[9]). We set

$$(1.7) \qquad \overline{V}(F) = \{s \in \Gamma(Q) | s = \pi(Y), Y \in V(F)\}$$

The $s \in \overline{V}(F)$ satisfies

$$(1.8) \qquad D_X s = 0$$

for any $X \in \Gamma(E)$ ([4], [9]).

Let $\Lambda^r(M)$ be the space of all r-forms on M. We have the decompositions of $\Lambda^r(M)$ and the exterior derivative d with respect to F:

$$(1.9) \qquad \Lambda^r(M) = \sum_{w+z=r} \Lambda^{w,z}(M),$$

$$(1.10) \qquad d = d' + d'' + d'''$$

([5], [14], [16], [18]). Let $\Delta^r(M)$ be a subspace of $\Lambda^{o,r}(M)$ composed of d'-closed (o,r)-forms, that is, the space of all basic (o,r)-forms on M ([5], [14]). An operator $\delta : \Lambda^r(M) \to \Lambda^{r-1}(M)$ is defined by

$$\delta = (-1)^{(p+q)(r+1)+1} * d*$$

where $*$ denotes the Hodge star operator. Then δ has a decomposition : $\delta = \delta' + \delta'' + \delta'''$. The operator δ'' is defined by

$$\delta'' = (-1)^{(p+q)(r+1)+1} * d''*$$

on $\Lambda^r(M)$ ([16], [18]). Let $\Delta_0^r(M)$ be the subspace of $\Delta^r(M)$ composed of forms with compact supports. Then the pre-Hilbert metric $\ll\, , \,\gg$ on $\Delta_0^r(M)$ is defined by

$$\ll \phi, \psi \gg = \int_M \phi \wedge *\psi$$

Let $\Omega^r(M,Q)$ (resp. $\Omega_0^r(M,Q)$) be the space of all Q-valued r-forms (resp. Q-valued r-forms with compact support) on M. On $\Omega_0^r(M,Q)$, we may introduce a global scalar product $\ll , \gg$ by

$$\ll t, u \gg = \int_M g_Q(t \wedge *u)$$

Let $\Gamma_0(Q)$ be the space of all sections of Q with compact supports and let $L^2(Q)$ be the completion of $\Gamma_0(Q)$ with respect to the global scalar product $\ll , \gg$.

Definition 1.1 ([19],[22]). An element $s \in L^2(Q) \cap \Gamma(Q)$ is called an *L^2-transversefield* of F.

Definition 1.2 ([23]). An operator $div_D : \Gamma(Q) \to C^\infty(M)$ defined by $div_D t = g^{ab} g_Q(D_{X_a} t, \pi(X_b))$ is called the *transverse divergence* with respect to D.

Definition 1.3 ([23]). The *transverse gradient* $grad_D f$ of a function f with respect to D is defined by $grad_D f = g^{ab} X_a(f) \pi(X_b)$

2. The *transverse Lie derivative* $\Theta(Y)$ with respect to $Y \in V(F)$ is defined by

$$\Theta(Y)s = \pi([Y, Y_s]) \tag{2.1}$$

for any $s \in \Gamma(Q)$ with $\pi(Y_s) = s$.

For $Y \in V(F)$, the operator $A_D(Y) : \Gamma(Q) \to \Gamma(Q)$ is defined by

$$A_D(Y)t = \Theta(Y)t - D_Y t \tag{2.2}$$

Then we have

$$A_D(Y)t = -D_{Y_t} \pi(Y) \tag{2.3}$$

where $t = \pi(Y_t)$. This shows that

(i) $A_D(Y)$ depends only on $s = \pi(Y)$.

(ii) $A_D(Y)$ is a linear operator of $\Gamma(Q)$.

Thus we can use $A_D(s)$ instead of $A_D(Y)$ ([4]).

Let $d_D : \Omega^r(M,Q) \to \Omega^{r+1}(M,Q)$ be the exterior differential operator and the $d^*D : \Omega(M,Q) \to \Omega^{r-1}(M,Q)$ be defined ([3]).

The *Laplacian* Δ_D acting on $\Omega^r(M,Q)$ is defined by

$$\Delta_D = d_D d_D^* + d_D^* d_D \tag{2.4}$$

An element of $\Gamma(Q)$ is regarded as an element of $\Omega^0(M,Q)$.

The bundle map $\pi : TM \to Q$ is an element of $\Omega^1(M,Q)$. The Q-valued bilinear form α on M is defined by

$$\alpha(X,Y) = -(D_X\pi)(\dot{Y}) \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ ([3]). Since $\alpha(X,Y) = \pi(\nabla_X Y)$ for any $X, Y \in \Gamma(E)$, α is called the *second fundemental form* of F ([3]).

The *tension field* τ of F is defined by

$$\tau = g^{ij}\alpha(X_i, X_j) \tag{2.6}$$

([3]). We remark that $\tau = d_D^*\pi \in \Gamma(Q)$.

The foliation F is said to be *minimal* if $\tau = 0$.

Let x_0 be a fixed point of M and $\rho(x)$ the geodesic distance from x_0 to $x \in M$.

We set

$$B(2k) = \{x \in M | \rho(x) \leq 2k\} \tag{2.7}$$

for any $k > 0$. We consider a function μ on R which satisfies the following properties:

$$0 \leq \mu(y) \leq 1 \text{ on } R$$

$$\mu(y) = 1 \text{ for } y \leq 1$$

$$\mu(y) = 0 \text{ for } y \geq 2.$$

We define a family $\{w_k\}$ of Lipschitz continuous functions on M:

$$w_k(x) = \mu(\rho(x)/k), \quad k = 1, 2, \cdots$$

for any $x \in M$. Then the family $\{w_k\}$ has the following properties:

$$\begin{aligned} 0 \leq w_k(x) \leq 1 &\text{ for any } x \in M \\ \text{supp } w_k &\subset B(2k) \\ w_k(x) = 1 &\text{ for any } x \in B(2k) \\ \lim_{k\to\infty} w_k &= 1 \\ |dw_k| \leq Ck^{-1} &\text{ almost everywhere on } M \end{aligned} \tag{2.8}$$

where C is a positive constant independent of k ([5], [18], [19], [20]). We remark that, for any $s \in L^2(Q) \cap \overline{V}(F), w_k s \to s$ as $k \to \infty$ in the strong sense.

We now introduce some lemmas for later use.

Lemma 2.1 ([22]). *For any $s \in \overline{V}(F)$, it holds that*

$$\|d'' w_k \otimes s\|^2_{B(2k)} \leq C^2 k^{-2} \|s\|^2_{B(2k)}$$

Lemma 2.2 ([1]). *If F is minimal, then*

$$\int_{B(2k)} div_D(w_k^2 s) dM = 0$$

for any $s \in \overline{V}(F)$.

Moreover, for any $s \in \overline{V}(F)$, we have

$$\begin{aligned} (2.9) \quad div_D((div_D s) w_k^2 s) &= 2g_Q((w_k div_D s)s, grad_D w_k) \\ &\quad + g_Q(w_k^2 s, grad_D div_D s) + (w_k div_D s)^2 \end{aligned}$$

$$(2.10) \qquad g_Q(grad_D div_D t, t) = \sigma(t)(div_D t)$$

([1], [23]).

By the direct calculation, we obtain

$$\begin{aligned} (2.11) \qquad div_D(A_D(s)(w_k^2 s)) &= 2w_k g_Q(D_{\sigma(s)}, grad_D w_k) \\ &\quad + w_k^2 div_D(D_{\sigma(s)} s) \end{aligned}$$

By means of Lemma 2.2 and (2.9)-(2.11), we have

Lemma 2.3. *If F is minimal, then it holds that*

$$\begin{aligned} \int_{B(2k)} [w_k^2 \{Ric(s) + Tr(A_D(s) A_D(s)) - (div_D s)^2\} \\ + 2w_k g_Q(D_{\sigma(s)} s - (div_D s), grad_D w_k)] dS = 0 \end{aligned}$$

for any $s \in \overline{V}(F)$, where $Ric_D(s) = g_Q(\rho_D(s), s)$ and dS denotes the volume element of $B(2k)$.

3. Let ${}^t A_D(s)$ be the transpose of $A_D(s)$, that is, ${}^t A_D(s)$ satisfies the following equality:

$$g_Q(A_D(s)t, u) = g_Q(t, {}^t A_D(s) u)$$

for any $t, u \in \Gamma(Q)$.

For $s \in \overline{V}(F)$, let $B_D(s) : \Gamma(Q) \to \Gamma(Q)$ be an operator defined by

$$B_D(s) = A_D(s) - {}^tA_D(s) \tag{3.1}$$

([11]). The operator $B_D(s)$ is skew-symmetric, that is,

$$g_Q(B_D(s)t, u) = -g_Q(t, B_D(s)u) \tag{3.2}$$

for any $t, u \in \Gamma(Q)$. Therefore, $T_r(B_D(s)) = 0$.

On the other hand, by the direct calculation, we get

$$T_r((B_D(s))^2) = 2Tr(A_D(s)A_D(s)) - 2Tr({}^tA_D(s)A_D(s)),$$

which together with Lemma 2.3 and the equality:

$$\int_{B(2k)} w_k^2 Tr({}^tA_D(s)A_D(s))dS = \ll w_k Ds, w_k Ds \gg_{B(2k)}$$

yields

$$\begin{aligned} &\frac{1}{2}\int_{B(2k)} \{Tr(w_k^{2t}B_D(s)B_D(s)) + (w_k div_D s)^2\}dS \\ &= \int_{B(2k)} \{w_k^2 g_Q(\rho_D(s) + \Delta_D(s), s) - \frac{1}{2}(w_k div_D s)^2 \\ &\quad -2(w_k div_D s) g_Q(s, grad_D w_k)\}dS \end{aligned} \tag{3.3}$$

because of (3.2).

By means of the Schwarz inequality for the local scalar product $< , >$, it holds that $|2(w_k div_D s)g_Q(s, grad_D w_k)| \le \frac{1}{2}(w_k div_D s)^2 + 2c^2k^{-2} < s, s >$ ([1]). The above inequality and (3.3) imply

$$\begin{aligned} &\frac{1}{2}\int_{B(2k)} \{Tr(w_k^{2t}B_D(s)B_D(s)) + (w_k div_D s)^2\}dS \\ &\le \int_{B(2k)} w_k^2 g_Q(\rho_D(s) + \Delta_D(s), s)dS + 2c^2k^{-2}\int_{B(2k)} < s, s > dS. \end{aligned} \tag{3.4}$$

Definition 3.1 ([11]). If $s \in \overline{V}(F)$ satisfies

$$B_D(s) = 0 \text{ and } div_D s = 0,$$

then s is called a *transverse harmonic field* of F.

Proof of Theorem A. Suppose that $\Delta_D(s) = -\rho_D(s)$. Since

$$2c^2k^{-2}\|s\|^2_{B(2k)} \to 0 \text{ as } k \to \infty,$$

we have from (3.4)

$$\begin{aligned} 0 &\le \frac{1}{2}\int_{B(2k)} \{Tr({}^tB_D(s)B_D(s)) + (div_D s)^2\}dS \\ &\le \int_{B(2k)} g_Q(\rho_D(s) + \Delta_D(s), s)dS. \end{aligned}$$

Therefore, we have $B_D(s) = 0$ and $div_D s = 0$, that is, s is L^2-transverse harmonic field. Conversely, if s is a transverse harmonic field, that is, $g_Q(A_D(s)(\pi(X_a)), \pi(X_b)) = g_Q(\pi(X_a), A_D(s)(\pi(X_b)))$ and $div_D s = 0$, then we obtain

$$\begin{aligned} 0 &= g_Q(D_{X_c}D_{X_a}s, \pi(X_b)) + g_Q(D_{X_a}s, D_{X_c}\pi(X_b)) \\ &\quad -g_Q(D_{X_c}\pi(X_a), D_{X_b}s) - g_Q(\pi(X_a), D_{X_c}D_{X_b}s). \end{aligned}$$

Transvecting g^{ac} to this equation, it follows that $\Delta_D(s) + \rho_D(s) = 0$ with the aid of $div_D s = 0$. This completes the proof of Theorem A.

Proof of Theorem B. Let $s \in \overline{V}(F)$ be an L^2-transverse harmonic field. Then Theorem A yields

$$\ll \rho_D(s) + \Delta_D(s), w_k^2 s \gg_{B(2k)} = 0.$$

Hence, if ρ_D is non-negative everywhere in M, then

$$\ll \Delta_D(s), w_k^2 s \gg_{B(2k)} \le 0 \tag{3.5}$$

On the other hand, for any $s \in \overline{V}(F)$, it holds that

$$\begin{aligned} \ll \Delta_D(s), w_k^2 s \gg_{B(2k)} &= \ll w_k Ds, w_k Ds \gg_{B(2k)} \\ &\quad +2 \ll w_k Ds, d''w_k \otimes s \gg_{B(2k)} \end{aligned}$$

and

$$|2 \ll w_k Ds, d''w_k \otimes s \gg_{B(2k)} | \le \frac{1}{2}\|w_k Ds\|^2_{B(2k)} + 2c^2k^{-2}\|s\|^2_{B(2k)},$$

which and (3.5) yield

$$\begin{aligned} &\|w_k Ds\|^2_{B(2k)} - \frac{1}{2}\|w_k Ds\|^2_{B(2k)} - 2c^2k^{-2}\|s\|^2_{B(2k)} \\ &\le \|w_k Ds\|^2_{B(2k)} + 2 \ll w_k Ds, d''w_k \otimes s \gg_{B(2k)} \\ &\le 0. \end{aligned}$$

Thus, as $k \to \infty$, we have

$$0 \leq \frac{1}{2}\|Ds\|^2_{B(2k)} \leq \ll \Delta_D(s), s \gg_{B(2k)} \leq 0,$$

and consequently, $Ds = 0$, that is s is D-parallel. Moreover, if the Ricci operator ρ_D is positive at least one point of M, then any transverse harmonic field s is zero, which completes the proof of Theorem B.

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After the submission of this paper, it came to our attension that the similar results were published in Ann. Global Anal. Geom., Vol. 7, No.1 (1989), 47-57 by S. Nishikawa and P. Tondeur. However, our results in this paper were obtained independently.

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