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SEMI-IDEMPOTENTS IN THE GROUP RING OF A CYCLIC GROUP OVER THE FIELD OF RATIONALS

W. B. Vasantha

Introduction

In [1] the author has introduced the notion of semi-idempotents to group rings. This paper is concerned with the study of semi-idempotents in QG, where Q is the field of rationals and G is a cyclic group of prime order p. Here a necessary and sufficient conditions are obtained for $\alpha = \alpha_0 + \alpha_1 g + \alpha_2 g^2 + \cdots + \alpha_{p-1} g^{p-1}$ to be a semi-idempotent in QG where $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_p$ are rationals in Q and $g^p = 1$. For definitions and results used please refer [1].

Proposition 1. *Let $G = < g/g^2 = 1 >$ be a cyclic group of order 2. Q be a rational field. QG the group ring of G over Q. Then $\alpha = \alpha_0 + \alpha_1 g$ is a semi-idempotent if and only if $\alpha_0 = \pm\alpha_1 = \frac{1}{2}$ or $\alpha_0 = 1 + \alpha_1$ or $\alpha_0 = 1 - \alpha_1$ and if*

$$\alpha \notin w(Q[G]) \text{ but } \alpha^2 - \alpha \in w(Q[G]).$$

Proof. Let $\alpha = \alpha_0 + \alpha_1 g$ be a semi-idempotent in QG. To obtain conditions on α_0 and α_1 where $\alpha_0, \alpha_1 \in Q$. Let P be the proper ideal generated by $(\alpha_0 + \alpha_1 g)^2 - (\alpha_0 + \alpha_1 g)$. Then

$$\begin{aligned}(\alpha_0 + \alpha_1 g)^2 - (\alpha_0 + \alpha_1 g) &= \alpha_0^2 + 2\alpha_0\alpha_1 g + \alpha_1^2 g^2 - \alpha_0 - \alpha_1 g \\ &= (\alpha_0^2 + \alpha_1^2 - \alpha_0) + (2\alpha_0\alpha_1 - \alpha_1) g \\ &= A_0 + A_1 g \text{ where } A_0, A_1 \in Q\end{aligned}$$

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with $A_0 = \alpha_0^2 + \alpha_1^2 - \alpha_0$ and $A_1 = 2\alpha_0\alpha_1 - \alpha_1$. Now $A_0 + A_1 g \in P$ implies

$$A_0 g + A_1, A_0^2 + A_0 A_1 g, A_0 A_1 g + A_1^2 \text{ are in } P.$$

Hence $(A_0^2 + A_0A_1g) - (A_0A_1g + A_1^2)$ is in P. Thus $A_0^2 - A_1^2 \in P$. This is possible if and only if $A_0^2 - A_1^2 = 0$. That is $(A_0 - A_1)(A_0 + A_1) = 0$. Since Q is the field of rationals either

(a) $A_0 = A_1$ and (or)

(b) $A_0 + A_1 = 0$.

(a) Suppose $A_0 = A_1$ then $\alpha_0^2 + \alpha_1^2 - \alpha_0 = 2\alpha_0\alpha_1 - \alpha_1$ that is

$$\begin{aligned} \alpha_0^2 + \alpha_1^2 - 2\alpha_0\alpha_1 &= \alpha_0 - \alpha_1 \\ (\alpha_0 - \alpha_1)^2 &= (\alpha_0 - \alpha_1) \\ (\alpha_0 - \alpha_1)[\alpha_0 - \alpha_1 - 1] &= 0 \end{aligned}$$

The two possibilities are

$$\begin{aligned} \alpha_0 &= \alpha_1 \\ \alpha_0 &= 1 + \alpha_1. \end{aligned}$$

If $\alpha_0 = \alpha_1$ we get $\alpha = \alpha_0 + \alpha_0 g = \alpha_0(1+g)$. Therefore P is generated by $[\alpha_0(1+g)]^2 - [\alpha_0(1+g)] = \alpha_0[\alpha_0(1+g)^2 - (1+g)]$. Cancelling α_0 as $P\alpha_0^{-1} = P$ we get P is generated by

$$\begin{aligned} \alpha_0 + \alpha_0 g^2 + 2\alpha_0 g - 1 - g &= \alpha_0 + \alpha_0 + 2\alpha_0 g - 1 - g \\ &= (2\alpha_0 - 1) + g(2\alpha_0 - 1) \\ &= (2\alpha_0 - 1)(1+g). \end{aligned}$$

So α will be a semi-idempotent only if $2\alpha_0 - 1 = 0$. For otherwise $\alpha = 1 + g \in P$. Thus $\alpha_0 = \frac{1}{2} = \alpha_1$. So if $\alpha_0 = \alpha_1 = \frac{1}{2}$ then α is a semi-idempotent. On the other hand suppose $\alpha_0 = 1 + \alpha_1$. Then

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 g \\ &= 1 + \alpha_1 + \alpha_1 g = 1 + \alpha_1(1+g). \end{aligned}$$

The ideal P is generated by

$$\begin{aligned} &\{1 + \alpha_1(1+g)\}^2 - \{1 + \alpha_1(1+g)\} \\ &= 1 + \alpha_1^2(1+g)^2 + 2\alpha_1(1+g) - 1 - \alpha_1(1+g) \\ &= 1 + \alpha_1^2 + 2\alpha_1^2 g + \alpha_1^2 + 2\alpha_1 + 2\alpha_1 g - 1 - \alpha_1 - \alpha_1 g \\ &= 2\alpha_1^2 + 2\alpha_1^2 g + \alpha_1 + \alpha_1 g \\ &= (2\alpha_1^2 + \alpha_1)(1+g). \end{aligned}$$

Thus $1+g \in P$. But $1+\alpha_1(1+g) \notin P$, if P is to be a proper ideal. Hence $\alpha = \alpha_0 + \alpha_1 g$ is a semi-idempotent if $\alpha_0 = 1 + \alpha_1$.

(b) Suppose $A_0 + A_1 = 0$ then

$$\begin{aligned} \alpha_0^2 + \alpha_1^2 - \alpha_0 + 2\alpha_0\alpha_1 - \alpha_1 &= 0 \\ (\alpha_0 + \alpha_1)^2 - (\alpha_0 + \alpha_1) &= 0 \\ (\alpha_0 + \alpha_1)[\alpha_0 + \alpha_1 - 1] &= 0. \end{aligned}$$

This forces $\alpha_0 + \alpha_1 = 0$ or $\alpha_0 + \alpha_1 = 1$ (Since Q is the field of rationals). If $\alpha_0 = -\alpha_1$ then $\alpha = \alpha_0 - \alpha_0 g$ that is $\alpha = \alpha_0(1-g)$. Now P is generated by

$$\{\alpha_0(1-g)\}^2 - \alpha_0(1-g) = \alpha_0[\alpha_0(1-g)^2 - (1-g)].$$

Thus P is generated by

$$\alpha_0(1-g)^2 - (1-g) = (2\alpha_0 - 1)(1-g).$$

So $\alpha \in P$ only if $2\alpha_0 - 1 \neq 0$ so the only possibility for α to be a semi-idempotent is that $\alpha_0 = \frac{1}{2}$. Thus $\alpha_0 = -\alpha_1 = \frac{1}{2}$ gives $\alpha = \alpha_0(1+g) = \frac{1}{2}(1+g)$. If $\alpha_0 + \alpha_1 = 1$, then $\alpha_0 = 1 - \alpha_1$

$$\begin{aligned} \alpha &= 1 - \alpha_1 + \alpha_1 g \\ &= 1 - \alpha_1(1-g). \end{aligned}$$

Now P is generated by

$$\begin{aligned} &[1 - \alpha_1(1-g)]^2 - (1 - \alpha_1(1-g)) \\ &= 1 - 2\alpha_1(1-g) + \alpha_1^2(1-g)^2 - 1 + \alpha_1(1-g) \\ &= (2\alpha_1^2 - \alpha_1)(1-g). \end{aligned}$$

Clearly $1 - g \in P$ but $1 - \alpha_1(1-g) \notin P$ if P is to be a proper ideal i.e. if $\alpha = 1 - \alpha_1(1-g)$ is to be a semi-idempotent. Thus $\alpha_0 = 1 - \alpha_1$ gives α to be a semi-idempotent. Clearly if $\alpha \notin w(Q[G])$ but $\alpha^2 - \alpha \in w(Q[G])$ then α is a semi-idempotent.

Conversely if $\alpha \notin w(Q[G])$ but $\alpha^2 - \alpha \in w(Q[G])$ then by definition of $w(Q[G])$ in [1] we have α to be a semi-idempotent.

Further if $\alpha_0 = \pm\alpha_1 = \frac{1}{2}$ then $\alpha = \frac{1}{2}(1 \pm g)$, clearly the ideal P generated by $\alpha^2 - \alpha$ does not contain $\frac{1}{2}(1 \pm g)$ as $P = 0$. Thus $\alpha = \frac{1}{2}(1 \pm g)$ is a semi-idempotent.

Clearly if $\alpha_0 = 1 + \alpha_1$ or $\alpha_0 = 1 - \alpha_1$, then $\alpha = \alpha_0 + \alpha_1 g$ does not belong to P, P generated by $\alpha^2 - \alpha$.

We shall prove a similar result for a cyclic group of order 3 and then generalize it for any prime p.

Proposition 2. *Let G be a cyclic group of order 3, $G = < g | g^3 = 1 >$. QG be the group ring of G over Q. Then $\alpha = \alpha_0 + \alpha_1 g + \alpha_2 g^2$ is a semi-idempotent in QG if and only if*

1. $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$
2. $\alpha_0 = 1 + \alpha_1$ *and* $\alpha_1 = \alpha_2$
3. $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

Proof. Let $\alpha = \alpha_0 + \alpha_1 g + \alpha_2 g^2$ be a semi-idempotent in QG. To obtain conditions on α_0, α_1 and α_2. Let P be the ideal generated by $\alpha^2 - \alpha$.

$$\begin{aligned}
\alpha^2 - \alpha &= (\alpha_0 + \alpha_1 g + \alpha_2 g^2)^2 - (\alpha_0 + \alpha_1 g + \alpha_2 g^2) \\
&= \alpha_0^2 + \alpha_1^2 g^2 + \alpha_2^2 g + 2\alpha_0\alpha_1 g + 2\alpha_0\alpha_2 g^2 \\
&\quad + 2\alpha_1\alpha_2 - \alpha_0 - \alpha_1 g - \alpha_2 g^2 \\
&= (\alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0) + (\alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1)g \\
&\quad + (\alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2)g^2 \\
&= A_0 + A_1 g + A_2 g^2 \text{ where}
\end{aligned}$$

$A_0, A_1, A_2 \in Q$ with

$$\begin{aligned}
A_0 &= \alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0 \\
A_1 &= \alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1 \\
A_3 &= \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2.
\end{aligned}$$

Now, $A_0 + A_1 g + A_2 g^2$, $A_0 g + A_1 g^2 + A_2$, $A_0 g^2 + A_1 + A_2 g$ are in P. Also $A_0^2 + A_0 A_1 g + A_0 A_2 g^2$, $A_2 A_0 g^2 + A_2 A_1 + A_2^2 g$ and $A_2 A_0 g^2 + A_2 A_1 + A_2^2 g$ are in P. So

$$\begin{aligned}
&(A_0^2 + A_0 A_1 g + A_0 A_2 g^2) - (A_2 A_0 g^2 + A_2 A_1 + A_2^2 g) \\
&\quad = (A_0^2 - A_1 A_2) + (A_0 A_1 - A_2^2)g \in P.
\end{aligned}$$

Similarly

$(A_2^2 - A_1 A_0) + (A_0 A_2 - A_1^2)g \in P$ and $(A_1^2 - A_0 A_2) + (A_1 A_2 - A_0^2)g \in P$.

Hence $(A_0^2 - A_1 A_2)(A_1 A_2 - A_0^2) - (A_1^2 - A_0 A_2)(A_0 A_1 - A_2^2)$ is in P. If P is to be a proper ideal

$$(A_0^2 - A_1 A_2)^2 = (A_1^2 - A_0 A_2)(A_2^2 - A_0 A_1)$$

Similarly

$$(A_1^2 - A_0A_2)^2 = (A_0^2 - A_1A_2)(A_2^2 - A_0A_1)$$
$$(A_2^2 - A_0A_1)^2 = (A_0^2 - A_1A_2)(A_1^2 - A_0A_2)$$

If $A_0^2 - A_1A_2 = 0$ then $A_1^2 - A_0A_2 = 0$ and $A_2^2 - A_0A_1 = 0$ If $A_0^2 - A_1A_2 \neq 0$ then $A_2^2 - A_0A_1 \neq 0$ and $A_1^2 - A_0A_1 \neq 0$ But $A_0^2 - A_1A_2 = \frac{(A_2^2 - A_0A_1)^2}{A_1^2 - A_0A_2}$ from the last equation. We get

$$(A_1^2 - A_0A_2)^2 = \frac{(A_2^2 - A_0A_1)^2 \times (A_2^2 - A_0A_1)}{A_1^2 - A_0A_2}.$$

So that

$$(A_1^2 - A_0A_2)^3 = (A_2^2 - A_0A_1)^3.$$

Since Q is the field of rationals we get

$$A_1^2 - A_0A_2 = A_2^2 - A_0A_1.$$

Similarly

$$A_1^2 - A_0A_2 = A_0^2 - A_1A_2.$$

and

$$A_0^2 - A_1A_2 = A_1^2 - A_0A_2.$$

Thus

$$A_1^2 - A_0A_2 = A_2^2 - A_0A_1 = A_0^2 - A_1A_2.$$

$$A_1^2 - A_0A_2 = A_2^2 - A_1A_0 \text{ implies } A_1^2 - A_2^2 = A_0A_2 - A_0A_1.$$

$$\begin{aligned}(A_1 - A_2)(A_1 + A_2) &= A_0(A_2 - A_1)\\ (A_1 - A_2)(A_1 + A_2) &= A_0(A_1 - A_2)\\ (A_1 + A_2 + A_0)(A_1 - A_2) &= 0.\end{aligned}$$

Since we are in P.

If $A_0 + A_1 + A_2 \neq 0$ we have

$$A_0 - A_1 = 0 \quad A_1 - A_2 = 0 \quad A_2 - A_0 = 0.$$

Thus

$$A_1 = A_0 = A_2 \tag{1}$$

or

$$A_0 + A_1 + A_2 = 0 \tag{2}$$

If

$$A_0^2 - A_1A_2 = 0 \quad A_1^2 - A_0A_2 = 0 \quad A_2^2 - A_0A_1 = 0$$

Then

$$A_0^2 = A_1A_2 \quad A_1^2 = A_0A_2 \text{ and } A_2^2 = A_0A_1.$$

If one of A_0, A_1 or A_2 is zero then

$$A_0 = A_1 = A_2 = 0. \tag{3}$$

If $A_0 \neq 0$ then $A_1 \neq 0$ and $A_2 \neq 0$. So $A_0 = \frac{A_1^2}{A_2}$ $A_2^2 = A_0A_1$, $A_0 = \frac{A_2^2}{A_1}$. Then $\frac{A_1^2}{A_2} = \frac{A_2^2}{A_1}$ so $A_1^3 = A_2^3$ since Q is the field of rationals $A_0 = A_1 = A_2$ which is (1). Suppose condition (1) is true then

$$\begin{aligned} \alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0 &= \alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1 \\ &= \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2 \\ \alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1 &= \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2 \\ \alpha_1^2 - \alpha_2^2 &= 2\alpha_0\alpha_1 - 2\alpha_0\alpha_2 + \alpha_2 - \alpha_1 \\ &= 2\alpha_0(\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2) \end{aligned}$$

$$(\alpha_1 - \alpha_2)[\alpha_1 + \alpha_2 - 2\alpha_0 + 1] = 0.$$

So $\alpha_1 = \alpha_2$ or $\alpha_1 + \alpha_2 - 2\alpha_0 + 1 = 0$. Let $\alpha_1 = \alpha_2$. Then

$$\begin{aligned} \alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0 &= \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2. \\ \alpha_0^2 + 2\alpha_1^2 - \alpha_0 &= \alpha_1^2 + 2\alpha_0\alpha_1 - \alpha_1 \\ \alpha_0^2 + \alpha_1^2 - 2\alpha_0\alpha_1 + \alpha_1 - \alpha_0 &= 0 \\ (\alpha_0 - \alpha_1)^2 - (\alpha_0 - \alpha_1) &= 0 \\ (\alpha_0 - \alpha_1)[\alpha_0 - \alpha_1 - 1] &= 0 \end{aligned}$$

The two possibilities are

$$\alpha_0 = \alpha_1 \text{ or } \alpha_0 = 1 + \alpha_1.$$

If $\alpha_0 = \alpha_1$, then $\alpha_0 = \alpha_1 = \alpha_2$. Hence $\alpha = \alpha_0(1 + g + g^2)$. So $\alpha^2 - \alpha = (3\alpha_0^2 - \alpha_0)(1 + g + g^2)$ $\quad 1 + g + g^2 \notin P$ only if $3\alpha_0^2 - \alpha_0 = 0$ i.e. $\alpha_0 = \frac{1}{3}$. So α is a semi-idempotent if $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$. So α is a semi-idempotent if

$$\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}.$$

Suppose $\alpha_1 = \alpha_2$ and $\alpha_0 = 1 + \alpha_1$. Then $\alpha = 1 + \alpha_1 + \alpha_1 g + \alpha_1 g^2 = 1 + \alpha_1(1 + g + g^2)$.
P is generated by $\alpha^2 - \alpha = (3\alpha_1^2 + \alpha_1)(1 + g + g^2)$. Since P is a proper ideal we have $1 + \alpha_1(1 + g + g^2) \notin P$. Hence α is a semi-idempotent of QG. Suppose $\alpha_1 + \alpha_2 = 2\alpha_0 - 1$, then $\alpha_0 = \frac{1+\alpha_1+\alpha_2}{2}$ substituting α_0 in the equation

$$\begin{aligned} \alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0 &= \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2 \\ &= \alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1 \end{aligned}$$

$$\begin{aligned} \frac{(1 + \alpha_1 + \alpha_2)^2}{4} + 2\alpha_1\alpha_2 - \frac{(1 + \alpha_1 + \alpha_2)}{2} &= \alpha_1^2 + \frac{2(1 + \alpha_1 + \alpha_2)}{2}\alpha_2 - \alpha_2 \\ &= \alpha_2^2 + \frac{2(1 + \alpha_1 + \alpha_2)}{2}\alpha_1 - \alpha_1 \end{aligned}$$

$$\frac{(1 + \alpha_1 + \alpha_2)^2}{4} + 2\alpha_1\alpha_2 - \frac{(1 + \alpha_1 + \alpha_2)}{2} = \alpha_1^2 + \frac{2\alpha_2(1 + \alpha_1 + \alpha_2)}{2} - \alpha_2$$

$$\begin{aligned} &\frac{1}{4} + \frac{\alpha_1^2}{4} + \frac{\alpha_2^2}{4} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_1\alpha_2}{2} + 2\alpha_1\alpha_2 - \frac{1}{2} - \frac{\alpha_1}{2} \\ &-\frac{\alpha_2}{2} = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2 + \alpha_2^2 - \alpha_2 \end{aligned}$$

$$\begin{aligned} \frac{1}{4} + \frac{3\alpha_1^2}{4} + \frac{3\alpha^2}{4} - \frac{3\alpha_1\alpha_2}{2} &= 0 \\ \frac{3}{4}(\alpha_1 - \alpha_2)^2 + \frac{1}{4} &= 0 \\ \frac{1}{4}[3(\alpha_1 - \alpha_2)^2 + 1] &= 0 \end{aligned}$$

$(\alpha_1 - \alpha_2)^2 = -\frac{1}{3}$ since we are in the field of rationals. This is impossible. So $\alpha_0 = \frac{1+\alpha_1+\alpha_2}{2}$ cannot occur. Suppose (2) is true

$$\begin{aligned} A_0 + A_1 + A_2 &= 0. \\ \alpha_0^2 + 2\alpha_1\alpha_2 - \alpha_0 + \alpha_2^2 + 2\alpha_0\alpha_1 - \alpha_1 + \alpha_1^2 + 2\alpha_0\alpha_2 - \alpha_2 &= 0 \\ (\alpha_0 + \alpha_1 + \alpha_2)^2 - (\alpha_0 + \alpha_1 + \alpha_2) = 0 & \\ \text{i.e. } (\alpha_0 + \alpha_1 + \alpha_2)[\alpha_0 + \alpha_1 + \alpha_2 - 1] = 0. & \end{aligned}$$

So $\alpha_0 + \alpha_1 + \alpha_2 = 0$ or $\alpha_0 + \alpha_1 + \alpha_2 = 1$.
If $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Then we have

$$\alpha \notin w(Q[G]) \text{ but } \alpha^2 - \alpha \in w(Q[G]).$$

So α is a semi-idempotent. If $\alpha_0 + \alpha_1 + \alpha_2 = 0$ then both α and $\alpha^2 - \alpha \in w(Q[G])$ hence α is not a semi-idempotent.

Conversely if (1), (2) or (3) is satisfied it can be easily verified that α is a semi-idempotent in QG.

Now we sketch. The proof of our main theorem, when G is a cyclic group of prime power order p.

Theorem 3. *Let G be a cyclic group of order p, p a prime. Q is the field of rationals. Then $\alpha = \alpha_0 + \alpha_1 g + \cdots + \alpha_{p-1} g^{p-1}$ is a semi-idempotent in QG if and only if*

(1) $\alpha_i = \alpha_j$ *for* $i = 1, 2, \cdots, p-1$ *and* $j = 1, 2, \cdots, p-1$ $\alpha_0 = 1 + \alpha_i$

(2) $\alpha_0 = \alpha_1 = \cdots = \alpha_{p-1} = \frac{1}{p}$

(3) $\sum_{i=0}^{p-1} \alpha_0 = 1.$

Proof. Let $\alpha = \alpha_0 + \alpha_1 + \alpha_1 g + \cdots + \alpha_{p-1} g^{p-1}$ be a semi-idempotent in QG To get conditions on $\alpha_0, \alpha_1, \cdots, \alpha_{p-1}$.

$$\begin{aligned}
\alpha^2 - \alpha &= (\alpha_0 + \alpha_1 g + \cdots + \alpha_{p-1} g^{p-1})^2 \\
&\quad -(\alpha_0 + \alpha_1 g + \cdots + \alpha_{p-1} g^{p-1}) \\
&= (\alpha_0^2 + 2\alpha_1\alpha_{p-1} + 2\alpha_2\alpha_{p-2} + \cdots + 2\alpha_r\alpha_{p-r} - \alpha_0) \\
&\quad +(\alpha_{\frac{p+1}{2}} + 2\alpha_0\alpha_1 + \cdots + 2\alpha_r\alpha_{p-r+1} - \alpha_1)g \\
&\quad + \cdots \\
&\quad + g^{p-1}(\alpha_j^2 + 2\alpha_0\alpha_{p-1} + \cdots + 2\alpha_{p-r}\alpha_{r-1} - \alpha_{p-1}) \\
&= A_0 + A_1 g + \cdots + A_{p-1} g^{p-1}
\end{aligned}$$

where

$$\begin{aligned}
A_0 &= \alpha_0^2 + 2\alpha_1\alpha_{p-1} + \cdots + 2\alpha_r 2_{p-r} - \alpha_0 \\
A_1 &= \alpha^2_{\frac{p+1}{2}} + 2\alpha_0\alpha_1 + \cdots + 2\alpha_r\alpha_{p-r+1} - \alpha_1 \\
\cdots &= \\
A_{p-1} &= \alpha_j^2 + 2\alpha_0\alpha_{p-1} + \cdots + 2\alpha_{p-r}\alpha_{r-1} - \alpha_{p-1}
\end{aligned}$$

where the suffix of A_i for all $i = 1, 2, \cdots, p-1$ is such that the sum of the suffixes of $\alpha_{j's}$, and $\alpha'_k s$ are equal to i mod p. We have $\beta = A_0 + A_1 g + \cdots + A_{p-1} g^{p-1} \in P$ then $\beta, \beta g, \beta g^2, \cdots, \beta g^{p-1}$ are in P. From these equations eliminate $g, g^2, \cdots, g^{p-1}$.
We get $A_0^p = A_1^p = \cdots = A_{p-1}^p$ or $A_0 + A_1 + \cdots + A_p = 0$. If $A_0 = A_1 = \cdots = A_{p-1}$ we get

$$\alpha_0 = 1 + \alpha_1 \text{ and } \alpha_1 = \alpha_2 = \cdots = \alpha_{p-1},$$

which makes α a semi-idempotent. If $\alpha_0 = \alpha_1 = \cdots = \alpha_{p-1} = \frac{1}{p}$ once again α is a semi-idempotent.
In case $A_0 + A_1 + \cdots + A_{p-1} = 0$, we have

$$(\alpha_0 + \alpha_1 + \cdots + \alpha_{p-1})^2 - (\alpha_0 + \cdots + \alpha_{p-1}) = 0$$

so that $\alpha_0 + \alpha_1 + \cdots + \alpha_{p-1} = 0$ or $\alpha_0 + \alpha_1 + \cdots + \alpha_{p-1} = 1$. If $\alpha_0 + \alpha_1 + \cdots + \alpha_{p-1} = 0$ then α is not a semi-idempotent. If $\alpha_0 + \alpha_1 + \cdots + \alpha_{p-1} = 1$ we have α to be a semi-idempotent.

Converse can be verified by direct calculations. Now we pose the following problem.

Problem. Let G be a cyclic group of prime order p. Q any field

(i) When is any $\alpha \in QG$ a semi-idempotent?

(ii) If p is not a prime, does QG have non-trivial idempotents other than the once characterized in [1].

References

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Department of Mathematics, Indian Institute of Technology, Madras-600 036, India