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ALMOST a-CONVEX QUASI-ORDERED SYNTOPOGENOUS SPACES

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1. Introduction

Burgess and Fitzpatrick [1] have introduced the concept of quasi-ordered syntopogenous spaces and it is known to be a nice generalization of topological ([12]), proximity ([14]) and uniform ([12], [13]) ordered spaces respectively. In [2], [3], [4] and [10], the authors have introduced various concepts of convex quasi-ordered syntopogenous spaces generalizing the classical one, but they fail to be a complete category except one in [4].

Let us denote by Qord (Syn) the topological category of quasi-ordered sets (syntopogenous spaces) and increasing (continuous, resp.) maps, and OSyn the mixed topological category of Syn and Qord ([9]).

The aim of the present paper is to introduce a notion of almost a-convex quasi-ordered syntopogenous spaces which generalizes a-convex spaces, and then we show that

1) the full subcategory a-ASyn of OSyn determined by almost a-convex spaces is bireflective in OSyn;

2) the full subcategory a-CSyn of OSyn determined by a-convex spaces is coreflective in a-ASyn and

3) the full subcategory SyOSyn of OSyn determined by symmetrizable spaces is bireflective in OSyn and hence SyOSyn is bireflective in p-ASyn.

It is assumed that the reader is familiar with the notion and results of Csaszar ([5]) and Matolcsy ([10]). We recall in particular that for an arbitrary quasi-ordered syntopogenous $(X, \mathcal{S}, \leq)$, among the increasing (decreasing) syntopogenous structures coarser than $\mathcal{S}$ there exists a finest one on $(X, \leq)$, and it is denoted by $\mathcal{S}^u(\mathcal{S}^\ell$, resp.) [2], [4], [10]). For

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an elementary operation a, a quasi-ordered syntopogenous space $(X,\mathcal{S},\leq)$ is called a-convex if $\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^\ell)^a$ ([2], [3], [4] and [10] for $a = i$ or p); $(X,\mathcal{S},\leq)$ is called symmetrizable if there exists a symmetrical i-convex syntopogenous structure $\mathcal{S}$ on $(X,\leq)$ such that $\mathcal{S}_0 < \mathcal{S} < \mathcal{S}_0^p$ ([10]); $(X,\mathcal{S},\leq)$ is called continuous if for $<\in \mathcal{S}$, there is $<_1\in \mathcal{S}$ such that $A < B$ implies $i(A) <_1 i(B)$ and $d(A) <_1 d(B)$ [11]; and $(X,\mathcal{S},\leq)$ is called feebly a-convex if for any $<\in \mathcal{S}$, there is a family $\varepsilon \subseteq \varepsilon_c$ such that $<\subseteq \varepsilon^{ta} < \mathcal{S}^a$ [4].

2. Almost a-convex spaces

Throught this section all spaces are assumed to be quasi-ordered syntopogenous spaces, and a will denote an elementary operation.

Definition 2.1. A space $(X,\mathcal{S},\leq)$ is said to be *almost a-convex* if $\mathcal{S} < (\mathcal{S}^u \vee \mathcal{S}^l)^a$.

From the definition, the following is immediate:

Remark 2.2. 1) A space $(X,\mathcal{S},\leq)$ is almost a-convex iff $\mathcal{S}_1 \vee \mathcal{S}_2 < \mathcal{S} < (\mathcal{S}_1 \vee \mathcal{S}_2)^a$, for some increasing (decreasing, resp.) syntopogenous structure $\mathcal{S}_1(\mathcal{S}_2$, resp.) on $(X,\leq)$.

2) Let a' be an elementary operation such that $\mathcal{S}^a < \mathcal{S}^{a'}$ for any syntopogenous structure $\mathcal{S}$. Then if $(X,\mathcal{S},\leq)$ is almost a-convex, then $(X,\mathcal{S},\leq)$ is almost a'-convex.

3) Every a-convex space is almost a-convex.

4) Every symmetrizable space is almost p-convex.

5) If $(X,\mathcal{S},\leq)$ is almost a-convex, then $(X,\mathcal{S}^a,\leq)$ is a-convex.

6) If $(X,\mathcal{S},\leq)$ is almost a-convex, then $(X,\mathcal{S},\leq)$ is feebly a-convex.

Lemma 2.3. *Suppose $(X,\mathcal{S},\leq)$ is a space satisfying $\mathcal{S}^c = \mathcal{S}^{cp}$. Then $(X,\mathcal{S},\leq)$ is symmetrizable iff $\mathcal{S}^{uc} < \mathcal{S} < \mathcal{S}^{usp}$.*

Proof. Suppose that $(X,\mathcal{S},\leq)$ is symmetrizable. Then there is a symmetrical i-convex syntopogenous structure $\mathcal{S}_0$ on $(X,\leq)$ such that $\mathcal{S}_0 < \mathcal{S} < \mathcal{S}_0^p$. Since $\mathcal{S}_0$ is symmetrical i-convex, $\mathcal{S}_0 = \mathcal{S}_0^{us}$, so that $\mathcal{S} < \mathcal{S}^{usp}$. Take any $<\in \mathcal{S}$ and let $A <^{uc} B$. Then $X - B < X - A$. Since $\mathcal{S} < \mathcal{S}_0^{usp}$, there is $<_0\in \mathcal{S}_0$ such that $<\subseteq<_0^{usp}$; hence $X - B <_0^{usp} X - A$. Using (4.7) and (3.35) of [5] it follows that for any $b \in X - B$, $A <_0^{us} X - b$. Since $\mathcal{S}_0^{us} < \mathcal{S}$, there is $<_1\in \mathcal{S}$ such that $<_0^{us}\subseteq<_1$, and hence $A <_1 X - b$ for all $b \in X - B$. Since $\mathcal{S}^c$ is perfect, $X - B <_1^c X - A$, or $A <_1 B$. Thus $\mathcal{S}^{uc} < \mathcal{S}$. The converse is trivial.

As the following two examples show, the statements of Remark 2.2 cannot be conversed except for 1) and 2).

Examples 2.4. 1) Let $(R, \leq)$ be the real line with the usual order $\leq$ and let $<$ be generated by the set of all closed sets in R. Obviousely, $<^u$ ($<^1$) is generated by $\{[r, \infty) : r \in R\}$ ($\{(-\infty, r] : r \in R\}$, resp.). Then for $a = p$ or b, $(<^u \cup >^l)^{qa} = \subseteq$ is the discrete syntopogenous structure on R, so that $(R, \{<\}, \leq)$ is almost a-convex. But it is neither $p(b)$-convex nor symmetrizable. In fact, for any $x \in R$, $x < x$. Thus $<^p = <^b = \subseteq$. Since $< \neq \subseteq$, it follows that $(R, \{<\}, \leq)$ is not $p(b)$-convex ([10]). Since $<^c$ is generated by the usual topology on R, $<^c$ is perfect. On the other hand, for each $r \in R$, $(-\infty, r) <^{uc} (-\infty, r)$, but $(-\infty, r) < (-\infty, r)$ is impossible. Thus by Lemma 2.3, $(R, \{<\}, \leq)$ is not symmetrizable.

2) Let $(R, \leq)$ be as above, and for each natural number n, $H_n = (-\infty, -n) \cup [n, \infty)$. Let us consider a symmetrical topogenous structure $\{<\}$ on R, where $A < B$ iff $A \subseteq B$ and there is some n such that $A \cap H_n = \emptyset$ implies $H_n \subseteq B$. Then $(R, \{<\}, \leq)$ is p-convex and weakly p-convex ([10]) $(R, \{<\}, \leq)$ is feebly p-convex ([4]). But $<^u = <^l = <_{\emptyset, R}$ is the indiscrete syntopogenous structure on R, and therefore it is not almost p-convex.

Proposition 2.5. *Suppose $(X, \mathcal{S}, \leq)$ is a space such that $\mathcal{S}^{ua} \sim \mathcal{S}^{au}$ and $\mathcal{S}^{la} \sim \mathcal{S}^{al}$. Then $(X, \mathcal{S}, \leq)$ is almost a-convex iff $(X, \mathcal{S}^a, \leq)$ is a-convex.*

Proof. Suppose $(X, \mathcal{S}^a, \leq)$ is a a-convex. Then we have $\mathcal{S} < \mathcal{S}^a \sim (\mathcal{S}^{au} \vee \mathcal{S}^{al})^a \sim (\mathcal{S}^{ua} \vee \mathcal{S}^{la})^a \sim (\mathcal{S}^u \vee \mathcal{S}^l)^a$ ([5]); hence $(X, \mathcal{S}, \leq)$ is almost a-convex. The converse is trivial by Remark 2.2.5.

It is well-known [11] that if $(X, \mathcal{S}, \leq)$ is a continuous space then $\mathcal{S}^{up} \sim \mathcal{S}^{pu}$ and $\mathcal{S}^{lp} \sim \mathcal{S}^{pl}$. Using this fact together with the above proposition, we have the following:

Corollary 2.6. *Let $(X, \mathcal{S}, \leq)$ be a continuous space. Then $(X, \mathcal{S}, \leq)$ is almost p-convex iff $(X, \mathcal{S}^p, \leq)$ is p-convex.*

Let us recall that an elementary operation a is said to be symmetrical iff $ac = ca$ ([5]).

Proposition 2.7. *For a symmetrical elementary operation a, $(X, \mathcal{S}, \leq)$ is almost a-convex iff $(X, \mathcal{S}^c, \leq)$ is almost a-convex. Moreover, if $(X, \mathcal{S}, \leq)$ is almost a-convex, then $(X, \mathcal{S}^s, \leq)$ is almost a-convex.*

Proof. The first part follows from the fact that $\mathcal{S} < (\mathcal{S}^u \vee \mathcal{S}^l)^a$ implies $\mathcal{S}^c < (\mathcal{S}^u \vee \mathcal{S}^l)^{ac} = (\mathcal{S}^u \vee \mathcal{S}^l)^{ca} = (\mathcal{S}^{uc} \vee \mathcal{S}^{lc})^a = (\mathcal{S}^{cl} \vee \mathcal{S}^{cu})^a$ ([4], [5]), and the second part from the fact $\mathcal{S}^s \sim (\mathcal{S} \vee \mathcal{S}^c)^q$.

Remark 2.8. Let $(X, \mathcal{S}, \leq)$ be a symmetrical space. Then the following statements are equivalent:

1) $(X, \mathcal{S}, \leq)$ is almost a-convex.

2) $\mathcal{S}_0 < \mathcal{S} < \mathcal{S}_0^{sa}$ for some increasing structure $\mathcal{S}_0$ on $(X, \leq)$.

3) $\mathcal{S}_0 < \mathcal{S} < \mathcal{S}_0^{sa}$ for some decreasing structure $\mathcal{S}_0$ on $(X, \leq)$.

Proof. Suppose $(X, \mathcal{S}, \leq)$ is almost a-convex, then by (2.6.2) of [4], one has $\mathcal{S} < (\mathcal{S}^u \vee \mathcal{S}^l)^a = (\mathcal{S}^u \vee \mathcal{S}^{uc})^a \sim \mathcal{S}^{usa}$. Clearly $\mathcal{S}^{us} < \mathcal{S}$. Hence $\mathcal{S}^u$ will do the job for $\mathcal{S}_0$. The other implications 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1) are trivial.

Proposition 2.9. *Let $(X, \mathcal{S}, \leq)$ be a compact symmetrical space. Then the following statements are equivalent:*

1) *$(X, \mathcal{S}, \leq)$ is almost p-convex.*

2) *$(X, \mathcal{S}, \leq)$ is symmetrizable.*

3) *$(X, \mathcal{S}, \leq)$ is almost i-convex.*

Proof. 1) and 2) are equivalent by Remark 2.8 and 2.2.4. Suppose $(X, \mathcal{S}, \leq)$ is almost p-convex. Then $\mathcal{S} < (\mathcal{S}^u \vee \mathcal{S}^l)^p$ and hence $\mathcal{S}^p \sim (\mathcal{S}^u \vee \mathcal{S}^l)^p$. Since $\mathcal{S}^u \vee \mathcal{S}^l < \mathcal{S}$, $\mathcal{S}^u \vee \mathcal{S}^l$ is compact. By Lemma 8 of [6], $\mathcal{S} < \mathcal{S}^u \vee \mathcal{S}^l$. Thus $\mathcal{S} \sim \mathcal{S}^u \vee \mathcal{S}^l$. Hence 1) implies 3). The converse is clear by Remark 2.2.2.

Lemma 2.10. *Let $G : \underline{C} \to \underline{D}$ be a functor and $\underline{A}(\underline{B})$ a subcategory of $\underline{C}(\underline{D}$, resp.) such that G has a restriction $E : \underline{A} \to \underline{B}$, i.e., $G \circ H = F \circ E$, where H, F are embedding functors. Suppose*

1) *$\underline{B}$ is a coreflective (reflective, resp.) subcategory of $\underline{D}$.*

2) *$E : \underline{A} \to \underline{B}$ is full and $G : \underline{C} \to \underline{D}$ is faithful and full; and for each $C \in \underline{C}$, there is some $A \in \underline{A}$ such that $E(A)$ is isomorphic to $\underline{B}$-coreflection ($\underline{B}$-reflection, resp.) of $G(C)$. Then $\underline{A}$ is a coreflective (reflective, resp.) subcategory of $\underline{C}$.*

Proof. Take any $C \in \underline{C}$. Let $u : F(B) \to G(C)$ be the $\underline{B}$-coreflection for $G(C)$, then by the assumption, there is an isomorphism $g : E(A) \to B$ for some $A \in \underline{A}$. Since G is full, there is an $h : H(A) \to C$ in $\underline{C}$ with $G(h) = u \circ F(g) : G \circ H(A) \to G(C)$. Then (A, h) is an $\underline{A}$-coreflection of C. Indeed, for any $f : H(R) \to C$ in $\underline{C}$, there is a unique $j : E(R) \to B$ in $\underline{B}$ such that the diagram

$$\begin{array}{ccc} F\circ E(R) & = & G\circ H(R) \\ & F(j)\downarrow & \searrow G(f) \\ & F(B) & \xrightarrow{u} G(C) \end{array}$$

commutes. Since $g : E(A) \to B$ is an isomorphism in $\underline{B}$, the diagram

$$\begin{array}{ccccc} G\circ H(R) & = & F\circ E(R) & & \\ F(g^{-1}\circ j)\downarrow & & & \searrow G(f) & \\ G\circ H(A) & = & F\circ E(A) & \xrightarrow{u\circ F(g)=G(h)} & G(C) \end{array}$$

commutes.
Since E is full, there is an $m : R \to A$ in $\underline{A}$ with $E(m) = g^{-1}\circ j$. Then $G\circ H(m) = F(g^{-1}\circ j)$. Since G is faithful, $h\circ H(m) = f$. Moreover, such an m is unique. Thus $\underline{A}$ is a coreflective subcategory of $\underline{C}$. The other statement follows by duality.

Let a-OSyn denote the full subcategory of OSyn consisting of those quasi-ordered syntopogenous space $(X, \mathcal{S}, \leq)$ with $\mathcal{S} \sim \mathcal{S}^a$.

Proposition 2.11. *a-OSyn is coreflective in OSyn.*

Proof. For any $(X, \mathcal{S}, \leq) \in$ *OSyn*, $(X, \mathcal{S}^a, \leq) \in a$-OSyn, and the identity map $1_X : (X, \mathcal{S}^a, \leq) \to (X, \mathcal{S}, \leq)$ is a continuous increasing map [5]. Let $(X, \mathcal{S}_0, \leq_0) \in a$-OSyn and $f : (Y, \mathcal{S}_0, \leq_0) \to (X, \mathcal{S}, \leq)$ a continuous increasing map. Let $g : (Y, \mathcal{S}_0, \leq_0) \to (X, \mathcal{S}^a, \leq)$ be the map f as set maps. Since $g^{-1}(\mathcal{S}) < \mathcal{S}_0$ and $(Y, \mathcal{S}_0, \leq_0) \in a$-OSyn, $g^{-1}(\mathcal{S}^a) < \mathcal{S}_0$. Thus g is a continuous increasing map. Thus $1_X : (X, \mathcal{S}^a, \leq) \to (X, \mathcal{S}, \leq)$ is the a-OSyn- coreflection of $(X, \mathcal{S}, \leq)$.

Let a-CSyn (a-WSyn, a-FSyn) denote the full subcategory of OSyn determined by a-convex (weakly a-convex, feebly a-convex, resp.) spaces. Then one has the following:

Corollary 2.12. 1) *a-CSyn is coreflective in a-ASyn.*
2) *a-WSyn is coreflective in a-FSyn.*

Proof. It is immediate from Proposition 2.11, and the fact that the commutative diagram

$$\begin{array}{ccccc} a\text{-}\underline{CSyn} & \longrightarrow & a\text{-}\underline{WSyn} & \longrightarrow & a\text{-}\underline{OSyn} \\ \downarrow & & \downarrow & & \downarrow \\ a\text{-}\underline{ASyn} & \longrightarrow & a\text{-}\underline{FSyn} & \longrightarrow & \underline{OSyn} \end{array}$$

clearly satisfies the assumption in Lemma 2.10, where the arrows denote embedding functors.

Theorem 2.13. a-ASyn is bireflective in OSyn.

Proof. Since OSyn is a properly fibred topological category, it is enough to show that a-ASyn is closed under the formation of initial sources in OSyn. Suppose $(f_i : (X, \mathcal{S}, \leq) \to (X_i, \mathcal{S}_i, \leq_i))_{i \in I}$ is an initial source in OSyn and each $(X_i, \mathcal{S}_i, \leq_i) \in a$-ASyn. Then $\mathcal{S} \sim \vee f_i^{-1}(\mathcal{S}_i)$ and $x \leq y$ iff $f_i(x) \leq_i f_i(y)$ for all $i \in I$. Since for each $i \in I$, $(X_i, \mathcal{S}_i, \leq_i) \in a$-ASyn, $\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2} < \mathcal{S}_i < (\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2})^a$, for some increasing(decreasing) syntopogenous structure $\mathcal{S}_{i_1}(\mathcal{S}_{i_2}$, resp.) on $(X_i, \leq_i)$. For each $i \in I$, $f_i^{-1}(\mathcal{S}_i) < f_i^{-1}((\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2})^a) = f_i^{-1}(\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2})^a$ ([5], (9.3), (9.7)). Thus $\vee f_i^{-1}(\mathcal{S}_i) < \vee f_i^{-1}(\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2})^a < (\vee f_i^{-1}(\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2}))^a = f_i^{-1}(\vee(\mathcal{S}_{i_1} \vee \mathcal{S}_{i_2}))^a$ $= (f_i^{-1}(\vee \mathcal{S}_{i_1}) \vee f_i^{-1}(\vee \mathcal{S}_{i_2}))^a$ ([5], (8.97), (8.98), (8.99), (9.10), (K_1)). From (1.1.5) and (1.1.6) of [10], $\vee f_i^{-1}(\mathcal{S}_{i_1})$ and $\vee f_i^{-1}(\mathcal{S}_{i_2})$ are increasing and decreasing on $(X, \leq)$, respectively. From (8.98) and (9.3) of [5], it follows that $\vee f_i^{-1}(\mathcal{S}_{i_1})$ and $\vee f_i^{-1}(\mathcal{S}_{i_2})$ are coarser than $f_i^{-1}(\mathcal{S}_i)$ for all $i \in I$. Hence by Remark 2.2.1, $(X, \mathcal{S}, \leq)$ is almost a-convex space.

Corollary 2.14. 1) *a-ASyn is topological and complete.*
2) *a-ASyn is closed under the formation of limits in OSyn.*

The following is now immediate from the above theorem, Lemma 2.10 and Theorem 3.3 in [4].

Corollary 2.15. 1) *a-CSyn is bireflective in a-OSyn.*
2) *a-WSyn is bireflective in a-OSyn.*

Let SyOSyn denote the full subcategory of OSyn determined by symmetrizable spaces. Then one has the following:

Proposition 2.16. *SyOSyn is bireflective in OSyn, and hence SyOSyn is bireflective in p-ASyn.*

Proof. Let us again show that SyOSyn is closed under the formation of initial sources in OSyn. Suppose $(f_i : (X, \mathcal{S}, \leq) \to (X_i \mathcal{S}_i, \leq_i))_{i \in I}$ is an initial source in OSyn and each $(X_i, \mathcal{S}_i, \leq_i)$ is symmetrizable. Since

$(X_i, \mathcal{S}_i, \leq_i) \in \underline{SyOSyn}$, there exists symmetrical i-convex syntopogenous structure $\mathcal{S}_{i_0}$ on $(X_i, \leq_i)$ such that $\mathcal{S}_{i_0} < \mathcal{S}_i < \mathcal{S}_{i_0}^p$. Then $\vee f_i^{-1}(\mathcal{S}_{i_0}) < \vee f_i^{-1}(\mathcal{S}_i) < \vee f_i^{-1}(\mathcal{S}_{i_0})^p < (\vee f_i^{-1}(\mathcal{S}_{i_0})^p)^p = (\vee f_i^{-1}(\mathcal{S}_{i_0}))^p$. Thus $(X, \mathcal{S}, \leq)$ is symmetrizable. Since SyOSyn is a full subcategory of p-ASyn, SyOSyn is also bireflective in p-ASyn.

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