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A WEAK PROJECTIVE COVER OF A MODULE

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1. Introduction

In [5], dualizing the notion of an injective envelope, Rotman defined a projective cover of a module and showed it is equivalent to the concept of already well-knowned one.

In [4], the first author showed that a well-knowned projective cover of a module implies the one in a sense of Rotman, but its converse is not always true.

In this paper, we introduce the concept of a weak projective cover of a module, which is same as a projective cover in a sense of Rotman. We have to investigate some properties of weak projective cover and find conditions under which two concepts are equivalent.

Throughout this paper, R denotes a ring with 1 and every module is a unitary left R-module. For terminology and notation, we refer to [3], [5].

2. Main results

We define a weak projective cover of a module, which is the dual concept of an injective envelope.

Definition. An epimorphism $\varepsilon : P \to M$ is a weak projective cover of a module M if P is a projective module and there exists an epimorphism dashed arrow below

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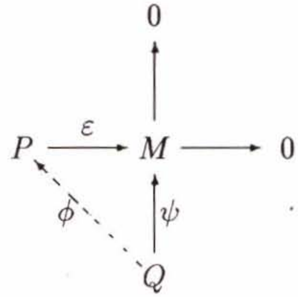
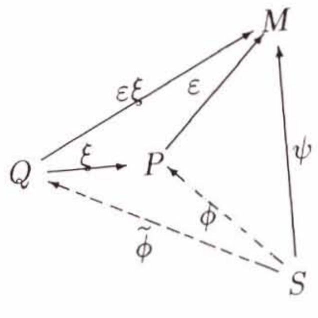


whenever Q is a projective module and $\psi : Q \to M$ is an epimorphism.

Remark. In [4], the first author showed that every projective cover of a module is a weak projective cover, but its converse is not always true. For example, let $\varepsilon : \mathbf{Z} \to \mathbf{Z}_2$ be the natural epimorphism as $\mathbf{Z}$-modules. Then it is not a projective cover but a weak projective cover of $\mathbf{Z}_2$.

Proposition 1. *Let $\xi : Q \to P$ be a weak projective cover of a projective module P and $\varepsilon : P \to M$ a homomorphism. Then $\varepsilon : P \to M$ is a weak projective cover if and only if $\varepsilon\xi : Q \to M$ is a weak projective cover.*

Proof. Consider the following diagram



where S is a projective module and $\psi : S \to M$ is an epimorphism.

Suppose that $\varepsilon : P \to M$ is weak projective cover. Then $\varepsilon\xi$ is an epimorphism and there exists an epimorphism $\phi : S \to P$ with $\varepsilon\phi = \psi$. Since $\varepsilon : Q \to P$ is a weak projective cover of P, there is an epimorphism $\widetilde{\phi} : S \to Q$ with $\xi\widetilde{\phi} = \phi$. Hence $\varepsilon\xi : Q \to M$ is a weak projective cover of M. Conversely, let $\varepsilon\xi : Q \to M$ be a weak projective cover of M. Then ε is an epimorphism and there exists an epimorphism $\widetilde{\phi} : S \to Q$ with

$(\varepsilon\xi)\widetilde{\phi} = \psi$. Let $\phi = \xi\widetilde{\phi}$. Then ϕ is epic and $\varepsilon\phi = \psi$. Hence $\varepsilon : P \to M$ is a weak projective cover.

Proposition 2. *Let $\varepsilon : P \to M$ be a weak projective cover of M and $\xi : M \to N$ a superfluous epimorphism. Then $\xi\varepsilon : P \to N$ is a weak projective cover of N.*

Proof. Let Q be a projective module and $\psi : Q \to N$ an epimorphism. Then there is an homomorphism $\widetilde{\psi} : Q \to M$ with $\xi\widetilde{\psi} = \psi$. Moreover, $M = \ker \xi + \mathrm{im}\ \widetilde{\psi}$. Since ξ is superfluous, $M = \mathrm{im}\ \widetilde{\psi}$. Hence $\widetilde{\psi}$ is epic. By assumption, there exists an epimorphism $\phi : Q \to P$ such that $\varepsilon\phi = \widetilde{\psi}$. It follows that $\xi\varepsilon : P \to N$ is a weak projective cover.

Theorem 3. *Let R be a IBN (=invariant basis number) ring such that every projective R-module is free. If a module M has a weak projective cover, then it is unique up to isomorphism.*

Proof. Let $\varepsilon : P \to M$ and $\xi : Q \to M$ be two weak projective covers of M. Then there are epimorphisms $\phi : P \to Q$ and $\psi : Q \to P$ such that $\xi\phi = \varepsilon$ and $\varepsilon\psi = \xi$. Let X and Y be bases of P and Q, respectively. Since ϕ is epic, $\phi(X)$ generates Q, and hence $|Y| \leq |\phi(X)| \leq |X|$. Similarly, $|X| \leq |Y|$. Thus $|X| = |Y|$. Since R is IBN, P and Q are isomorphic.

Remark. It is well-known that a projective cover of a module is unique. However, a weak projective cover of a module need not be unique in general.

Corollary 4. *Let R be a commutative ring such that every projective R-module is free. Then a module M has a unique weak projective cover if it has one.*

Example. Let R_1 be a quasi-local ring, R_2 a P.I.D., R_3 a Bézout ring, and $R_4 = K[x_1, x_2, \cdots, x_n]$, where K is a field. Then every R_i-module, $i = 1, 2, 3, 4$ has the unique projective cover if it has one.

Theorem 5. *R be a IBN ring such that every projective R-module is free. If M has a projective cover and $\varepsilon : P \to M$ is a weak projective cover of M, then it is the projective cover.*

Proof. Let $\xi : Q \to M$ be a projective cover of M. Then it is also a weak projective cover. By Theorem 3, there is an isomorphism $\phi : P \to Q$ with $\xi\phi = \varepsilon$. We claim that $\ker \varepsilon$ is superfluous in P. Let N be a submodule of P such that $\ker \varepsilon + N = P$. Since ϕ is an isomorphism, it follows that $Q = \ker \xi + \phi(N)$. Hence $Q = \phi(N)$, that is, $N = P$. Thus $\ker \varepsilon$ is superfluous in P.

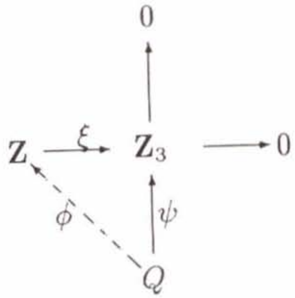
Corollary 6. *Ler R be a left perfect IBN ring such that every projective R-module is free. If $\varepsilon : P \to M$ is a weak projective cover of M, then it is the projective cover.*

Remark. This may be false without the hypothesis of left perfectness. For example, the natural map $\varepsilon : \mathbf{Z} \to \mathbf{Z}_2$ is a weak projective cover of $\mathbf{Z}_2$, but it not the projective cover.

Corollary 7. *Let R be a left perfect IBN ring such that every projective R-module is free. Then a direct sum of any weak projective covers is also a weak projective cover of the direct sum of modules.*

Remark. In general, a direct sum of weak projective covers is not a weak projective cover of the direct sum of modules.

For example, let $\xi : \mathbf{Z} \to \mathbf{Z}_3$ be the natural epimorphism. We show that it is weak projective cover of $\mathbf{Z}_3$. Consider the following diagram.



where Q is a projective module over $\mathbf{Z}$ and ψ an epimorphism. Since Q is projective, we may assume that $Q = \amalg_\alpha \mathbf{Z}_\alpha$, where $\mathbf{Z}_\alpha = \mathbf{Z}$ for each α. Let $u_\alpha : \mathbf{Z}_\alpha \to \amalg_\alpha \mathbf{Z}_\alpha$ be the αth injection. Since ψ is epic, there exists α such that $\psi u_\alpha = \xi$ or there is β such that $\psi u_\beta = 2\xi$. For each α, define $\phi_\alpha : \mathbf{Z}_\alpha \to \mathbf{Z}$ as follows :

$$\phi_\alpha(1) = \begin{cases} 1 & \text{if } \psi u_\alpha = \xi \\ -1 & \text{if } \psi u_\alpha = 2\xi \\ 0 & \text{if } \psi u_\alpha = 0. \end{cases}$$

Then $\xi\phi_\alpha = \psi u_\alpha$ for each α. Let ϕ be the coproduct map of the family $\{\phi_\alpha\}$. Then the existence of α or β implies ϕ is an epimorphism. Moreover $\xi\phi u_\alpha = \xi\phi_\alpha = \psi u_\alpha$ for each α. We have thus $\xi\phi = \psi$. So $\xi : \mathbf{Z} \to \mathbf{Z}_3$ is a weak projective cover of $\mathbf{Z}_3$. We already showed that $\varepsilon : \mathbf{Z} \to \mathbf{Z}_2$

is a weak projective cover of $\mathbf{Z}_2$. However, $\varepsilon \times \xi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}_2 \times \mathbf{Z}_3$ is not a weak projective cover of $\mathbf{Z}_2 \times \mathbf{Z}_3$. Indeed, let $\psi : \mathbf{Z} \to \mathbf{Z}_2 \times \mathbf{Z}_3$ be defined by $\psi(1) = (1,1)$. Then it is an epimorphism. But there exist no epimorphisms from $\mathbf{Z}$ to $\mathbf{Z} \times \mathbf{Z}$.

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