Kyungpook Mathematical Journal
Volume 31, Number 2, December, 1991

ON THE LIMIT-POINT CLASSIFICATION OF A WEIGHTED FOURTH ORDER DIFFERENTIAL EXPRESSIONS WITH COMPLEX COEFFICIENTS

M. F. EL-Zayat

1. Introduction

In this paper, we are concerned with the differential expression:

$$\begin{aligned} L(y) &\equiv (p_0(x)y^{(2)})^{(2)} + \frac{i}{2}\{(p_3(x)y')'' + p_3(x)y'')'\} \\ &\quad +(p_2(x)y')' + \frac{i}{2}\{(p_1(x)y)' + p_1(x)y'\} + p_4(x)y \qquad (1.1) \\ &= \lambda h(x)y \end{aligned}$$

on $[a,\infty)$, we assume that the coefficients function $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x)$ and $p_4(x)$ satisfy the following conditions:

i. $p_0(x), p_1(x), p_2(x), p_3(x)$ and $p_4(x)$ are real-valued and $p_0(x) > 0$
ii. $p_0'(x), p_3(x), p_2(x)$ and $p_1(x) \in AC_{loc}[a,\infty)$
iii. $p_4(x) \in L_{loc}[a,\infty)$. (1.2)
iv. The weight function $h(x) > 0$ and $h'(x)$ is continuous on $[a,\infty)$.

let N_+, N_- denote the number of linearly independent solutions of the differential equation;

$$L(y) = \lambda hy, \lambda = \mu + i\nu \qquad (1.3)$$

Received November 13, 1989.

For $\mu > 0$ and $\nu < 0$ respectively, the numbers N_+, N_- are known to be independent of λ in the respective half-planes.

Also, they satisfy the inequality $2 \leq N_+, N_- \leq 4$ (see 2, section 6.2). In the real-valued case [in (1.1) this corresponds to taking both $p_1(x)$ and $p_3(x)$ to be null on $[a, \infty)$, we have $N_+ = N_-$, in general this is not true in the complex case]. Furthere the integers (N_+, N_-) are also the deficiency indicies of the minimal closed symmetric operator that can be generated from $L_{(\cdot)}$ in the Hilbert function space $L^2_h[a, \infty)$ (see 1 definition XII 4.9).

In the general complex fourth-order case there are known to be exactly five possible cases for the deficiency indicies (See 2; section 6.2), i.e (2.2), (2.3), (3.2), (3.3) and (4.4)., but in this paper, we shall only be concerned with the minimal case (2.2).

2. Preliminaries

For (1.1), the quasi-derivatives $y^{[i]}$ are defined by:

$$\begin{aligned} y^{[0]} &= y, \quad y^{[1]} = y', \quad y^{[2]} = p_0(x)y'' + \frac{i}{2}(p_3(x)y') \\ y^{[3]} &= (y^{[2]})' + p_2(x)y' + \frac{i}{2}(p_3(x)y'' + p_1(x)y), \qquad (2.1) \\ y^{[4]} &= \lambda h(x)y \end{aligned}$$

The equation $L(y) = \lambda hy$, has the vector formulation:

$$Y' = AY \qquad (2.2)$$

where

$$Y = \begin{bmatrix} y \\ y' \\ y^{[2]} \\ y^{[3]} \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{ip_3}{2p_0} & \frac{1}{p_0} & 0 \\ -\frac{ip_1}{2} & -(p_2 + p_3^2/4p_0) & -\frac{ip_3}{2p_0} & 1 \\ (\lambda h - p_4) & -\frac{ip_1}{2} & 0 & 0 \end{bmatrix}$$

The Lagrange identity for (1.1) is :

$$L(y)\bar{z} - y\overline{L(z)} = [y, z]'$$

where

$$[y, z] = y'\bar{z}^{[2]} - y^{[2]}\bar{z}' + y^{[3]}\bar{z} - y\bar{z}^{[3]} \qquad (2.3)$$

For (1.1), we have the quadratic expression;

$$\begin{aligned}\{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}' &= (\lambda h - p_4)|y|^2 - p_0|y''|^2 + p_2|y'|^2 \\ &\quad + \frac{ip_3}{2}[y''\bar{y}' - y'\bar{y}''] + \frac{ip_1}{2}[y\bar{y}' - y'\bar{y}] \qquad (2.4)\end{aligned}$$

We transform Y by the transformation $W = MY$, where M is the diagonal matrix; $M =$ diagonal $\{\rho h^{\frac{3}{8}}, \rho^3 h^{\frac{1}{8}}, \frac{\rho^5}{p_0 h^{\frac{1}{8}}}, \frac{\rho^7}{p_0 h^{\frac{3}{8}}}\}$ and ρ is a positive twice continuously differentiable function. Clearly, the vector $W = [w_1, w_2, w_3, w_4]^T$ is

$$W = \left[\rho h^{\frac{3}{3}} y, \rho^3 h^{\frac{1}{8}} y', \frac{\rho^5}{p_0 h^{\frac{1}{8}}} y^{[2]}, \frac{\rho^7}{p_0 h^{\frac{3}{8}}} y^{[3]}\right]^T.$$

The vector W satisfies:

$$W' = (\frac{h^{\frac{1}{4}}}{\rho^2})CW \qquad (2.5)$$

$$C = \frac{\rho^2}{h^{\frac{1}{4}}}[MAM^{-1} + M'M^{-1}]$$

We consider the conditions:

$$\begin{aligned}&\text{i.} \quad \frac{|p_i(x)|^{2(4-i)}}{p_0(x)h^{1-\frac{i}{4}}} \text{ are } 0(1) \text{ as } x \to \infty, \quad (i = 1, 2, 3). \\ &\text{ii.} \quad \frac{-p_4(x)\rho^8}{p_0(x)h(x)} < k, \quad \text{for some } k > 0 \qquad (2.6) \\ &\text{iii.} \quad \frac{\rho^2}{h^{\frac{1}{4}}}\left[\frac{\rho'(x)}{\rho(x)} + \frac{p_0'(a)}{p_0(x)} + \frac{h'(x)}{h(x)}\right] = 0(1) \\ &\qquad \text{as } x \to \infty\end{aligned}$$

Calculation shows that $C = [C_{ij}]$ satisfies:

$$\begin{aligned}C_{i,i+1} &= 1, C_{ii} \text{ is bounded } (i = 1, 2, 3, 4) \text{ [by (2.6)]} \\ C_{32} &= -(p_2(x) + \frac{p_3^2(x)}{4p_0(x)})\frac{\rho^4}{p_0(x)h^{\frac{1}{2}}} = 0(1) \\ &\qquad \text{as } x \to \infty; \\ C_{31} &= C_{42} = -\frac{p_1\rho^6}{2p_0 h^{\frac{3}{4}}} = 0(1) \quad x \to \infty \\ C_{41} &= (\lambda h - p_4)\frac{\rho^8}{p_0 h} \geq -K \text{ [by (1.10)]},\end{aligned}$$

and otherwise $C_{ij} = 0$. From (2.5), we get:

$$\begin{aligned} w_r' &= \sum_{k=1}^{4} C_{rk} \frac{\rho^{2k-3}}{p_0^{\alpha k}} h^{(\frac{7-2k}{8})} y^{[k-1]}, \text{ where} \\ \alpha_k &= \delta_{3k} + \delta_{4k} \text{ (kronical delta)} \end{aligned} \tag{2.7}$$

we take, for $k = 1, 2, 3, 4$.

$$\sqrt{\frac{h^{\frac{1}{4}}}{\rho^2}} w_k = (h^{\frac{3-k}{2}} \rho^{4(k-1)})^{\frac{1}{2}} \frac{y^{[k-1]}}{p_0^{\alpha_k}}$$

If we define:

$$W_k = \int_a^t \frac{h^{\frac{1}{4}}}{\rho^2} |w_k|^2 ds$$

Thus:

$$W_k = \int_a^t \frac{\rho^{4(k-1)} h^{\frac{3-k}{2}}}{p_0^{2\alpha_k}} |y^{[k-1]}|^2 ds \tag{2.8}$$

3. Inequalities for a System of Equations

In this section we establish inequalities which will be used in the remainder of the paper.

Lemma 3.1. *If W is a solution of (2.5), such that C_{ij} are bounded for all i and j and if (2.8) holds, such that $W_1(\infty) < \infty$. Then W_4 and w_4 are $0(1)$. For $i = 1, 2, 3, 4$, we have:*

$$\begin{aligned} W_i &= 0(W_{i+1}^{1-\frac{1}{i}}) \text{ as } x \to \infty, \quad i = 1, 2, 3, \\ w_i^2 &= 0(W_{i+1}^{1-\frac{1}{2i}}); i = 1, 2, 3 \text{ as } x \to \infty \end{aligned}$$

Proof. Since $W_1(\infty) < \infty$, $W_1(x) = 0(1)$ as $x \to \infty$, i.e

$$W_1 = 0(W_2^{\frac{0}{1}}) \text{ as } x \to \infty.$$

For w_k, we write generally that:

$$w_k^2|_a^t = 2 \int_a^t w_k w_k' ds \tag{3.1}$$

Thus for $k = 1$, [by (2.7)], we get

$$\begin{aligned} w_1^2|_a^t &= 2\int_a^t [(\rho h^{\frac{3}{8}})'y + \rho h^{\frac{3}{8}} y']\rho h^{\frac{3}{8}} y ds \\ &= 2\int_a^t (\rho h^{\frac{3}{8}})'\rho h^{\frac{3}{8}} |y|^2 ds + 2\int_a^t \rho^2 h^{\frac{3}{4}} y' y ds \end{aligned}$$

The first integral of the right-hand side is $O(W_1)$ [by (2.6) and (2.8)] as $t \to \infty$ and by using cauchy Schwartz inequality, the second integral is :

$$\begin{aligned} \int_a^t \rho^2 h^{\frac{3}{4}} yy' ds &= 0([\int_a^t h|y|^2 ds]^{\frac{1}{2}} \cdot [\int_a^t \rho^4 h^{\frac{1}{2}} |y'|^2 ds]^{\frac{1}{2}}) \\ &= 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) \text{ as } t \to \infty. \end{aligned}$$

Thus, we can write that:

$$w_1^2|_a^t = 0(W_1^{\frac{1}{2}}[W_1^{\frac{1}{2}} + W_2^{\frac{1}{2}}]) \text{ as } t \to \infty$$

since $W_1(x) = 0(1)$ as $x \to \infty$. Then:

$$w_1^2 = 0(W_2^{\frac{1}{2}}) \text{ as } x \to \infty.$$

Next, for $k = 2$, we get:

$$W_2 = \int_a^t \frac{h^{\frac{1}{4}}}{\rho^2} w_2^2 ds$$

By (2.7), we get:

$$W_2 = \int_a^t w_2 w_1' ds - \int_a^t (\rho h^{\frac{3}{8}})' \rho^3 h^{\frac{1}{8}} yy' ds$$

Integrating by parts the first integral, we get:

$$W_2 = w_2 w_1|_a^t - \int_a^t w_1 w_2' ds - \int_a^t (\rho h^{\frac{3}{8}})' \rho^3 h^{\frac{1}{8}} yy' ds$$

By (2.7) and using Cauchy-Schwartz inequality we get:

$$W_2 = w_2 w_1|_a^t + 0(W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}) \text{ as } t \to \infty$$

But, for $k = 2$

$$w_2^2|_a^t = 2\int_a^t w_2 w_2' ds$$

By (2.7) and using Cauchy-Schwartz inequality we get:

$$w_2^2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}) \text{ as } t \to \infty$$

while

$$w_1 = 0(W_2^{\frac{1}{4}}) \text{ as } t \to \infty$$

Hence

$$w_1 w_2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]^{\frac{1}{2}}) \text{ as } t \to \infty$$

Then

$$W_2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]^{\frac{1}{2}}) + 0(W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}) \text{ as } t \to \infty$$

Thus, a division by W_2 gives:

$$\begin{aligned} 1 &= 0(W_2^{-\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]^{\frac{1}{2}}) + 0(W_2^{-\frac{1}{2}} + I) \text{ as } t \to \infty \\ &= 0(W_2^{-\frac{1}{2}} + I) \text{ as } t \to \infty, \end{aligned}$$

where $I = \frac{W_3^{\frac{1}{2}}}{W_2}$

This means that $I(t)$ is bounded above, moreover;

$$\lim_{t\to\infty} Inf I(t) > 0$$

Hence

$$W_2 = 0(W_3^{\frac{1}{2}}) \text{ as } t \to \infty$$

Further, return to w_2^2, we get:

$$w_2^2 = 0(W_3^{\frac{1}{4}}[W_3^{\frac{1}{4}} + W_3^{\frac{1}{2}}]) = 0(W_3^{\frac{3}{4}}) \text{ as } t \to \infty$$

i.e.

$$w_2 = 0(W_3^{\frac{3}{8}}) \text{ as } t \to \infty$$

For $k = 3$

$$W_3 = \int_a^t \frac{h^{\frac{1}{4}}}{\rho^2} w_3^2 ds$$

By (2.7), we get:

$$\begin{aligned} W_3 &= \int_a^t w_3 w_2' ds - \int_a^t [(\rho^3 h^{\frac{1}{8}})' - \frac{i p_3}{2p_0}\rho^3 h^{\frac{1}{8}}] y' \cdot \frac{\rho^5}{p_0 h^{\frac{1}{8}}} y' y^{[2]} ds \\ &= w_3 w_2 |_a^t - \int_a^t w_2 w_3' ds - \int_a^t (\rho^3 h^{\frac{1}{8}})' \frac{\rho^5}{p_0 h^{\frac{1}{8}}} y' y^{[2]} ds \\ &\quad + \frac{i}{2} \int_a^t \frac{p_3}{2p_0} \rho^8 y' y^{[2]} ds \end{aligned}$$

By (2.6) and using Cauchy-Schwartz inequality we get:

$$W_3 = w_2 w_3|_a^t + 0(W_3^{\frac{3}{4}} + W_3^{\frac{1}{4}} W_4^{\frac{1}{4}}) \text{ as } t \to \infty,$$

For, $k = 3$

$$w_3^2|_a^t = 2\int_a^t w_3 w_3' ds$$

By (2.7) and using Cauchy-Schwartz inequality, we get:

$$\begin{aligned} w_3^2 &= 0(W_3^{\frac{1}{2}}[W_3^{\frac{1}{2}} + W_4^{\frac{1}{2}}]) \text{ as } t \to \infty \\ w_3 &= 0(W_3^{\frac{1}{4}}[W_3^{\frac{1}{2}} + W_4^{\frac{1}{2}}]^{\frac{1}{2}}) \text{ as } t \to \infty \end{aligned}$$

while

$$w_2 = 0(W_3^{\frac{3}{8}}) \text{ as } t \to \infty.$$

Hence

$$w_2 w_3 = 0(W_3^{\frac{3}{8}}[W_3^{\frac{1}{2}} + W_4^{\frac{1}{2}}]^{\frac{1}{2}}) \text{ as } t \to \infty$$

i.e

$$w_3 = 0(W_3^{\frac{5}{8}}[W_3^{\frac{1}{2}} + W_4^{\frac{1}{2}}]^{\frac{1}{2}}) + 0(W_3^{\frac{1}{4}}[W_3^{\frac{1}{2}} + W_4^{\frac{1}{2}}]) \text{ as } t \to \infty$$

Thus, a division by W_3 gives:

$$1 = 0(W_3^{-\frac{1}{4}} + I) \text{ as } t \to \infty,$$

where $I = \frac{W_4^{\frac{1}{2}}}{W_3^{\frac{3}{4}}}$, this means that $I(t)$ is bounded above moreover

$$\lim_{t \to \infty} \inf I(t) > 0.$$

Hence

$$W_3 = 0(W_4^{\frac{2}{3}}) \text{ as } t \to \infty$$

Further, return to w_3^2, we get;

$$w_3^2 = 0(W_4^{\frac{1}{3}}[W_4^{\frac{1}{3}} + W_4^{\frac{1}{2}}]) = 0(W_4^{\frac{5}{6}}) \text{ as } t \to \infty$$

i.e

$$w_3 = 0(W_4^{\frac{5}{12}}) \text{ as } t \to \infty.$$

In a similar way, one can prove that as $t \to \infty$;

$$w_4 = 0(W_4^{\frac{11}{12}}), \text{ i.e. } W_4 = 0\ (1) \text{ as } t \to \infty$$

and consequently,

$$w_4 = 0(1), \text{ as } t \to \infty.$$

This completes the proof.
We also make use of the vector spaces;

$$\begin{aligned} V_+ &= \{y : My = \lambda hy\}\lambda \cap L_h^2[a,\infty) \\ V_- &= \{z : Mz = \bar{\lambda}h\bar{z}\} \cap L_h^2[a,\infty) \end{aligned}$$

Next define.

$$\begin{aligned} J_1(t) &= \int_a^t \frac{\rho^8|y^{[2]}|^2}{p_0^2} ds, \quad y \in V_+; \\ J_2(t) &= \int_a^t \frac{\rho^8|z^{[2]}|^2}{p_0^2} ds, \quad z \in V_-; \end{aligned}$$

4. Auxiliary Lemmas

To prove the theorem (4.1), we need the following two lemmas:

Lemma 4.1. *If* $\dim V_+ + \dim V_- > 4$, *then there is a* $y \in V_+$ *and* $z \in V_-$, *such that:*

$$[y, z] = 1$$

For the proof see (3).

Lemma 4.2. *Let* F *be a non-negative continuous function on* $[a, \infty)$ *and define*

$$H(t) = \int_a^t (t-s)^2 F(S) ds.$$

If as $t \to \infty$, $H(t) = 0(t^2[H'']^{\frac{3}{4}})$, *then*

$$\int_a^t F(s) ds = 0(1), \text{ as } t \to \infty.$$

For the proof see (3).

Theorem 4.1. *Let all the conditions of (1.2), (2.6) be satisfied and*

$$\int_0^\infty \frac{h^{\frac{1}{4}}\rho^6}{p_0} ds = \infty \tag{4.1}$$

Then the equation $M(y) = \lambda hy$, has at most two linearly independent solutions in $L_h^2[a,\infty)$.

Proof. We first show $J_1(\infty) < \infty$, from the quadratic expression we have:

$$\int_a^t \{(\lambda h - p_4)|y|^2 - p_0|y''|^2 + p_2|y'|^2 + \frac{ip_3}{2}(y''\bar{y}' - y'\bar{y}'') + \frac{ip_1}{2}(y\bar{y}' - y'\bar{y})\}(1-\frac{s}{t})^2\frac{\rho^8}{p_0}ds \cdot \tag{4.2}$$
$$= \int_a^t \{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}'(1-\frac{s}{t})^2\frac{\rho^8}{p_0}ds$$

Integrating by parts the right hand side, we have ;

$$\int_a^t \{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}'(1-\frac{s}{t})^2\frac{\rho^8}{p_0}ds = 0(1) - \int_a^t \{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}[(1-\frac{s}{t})^2\frac{\rho^8}{p_0}]'ds$$

But

$$[(1-\frac{s}{t})^2\frac{\rho^8}{p_0}]' = 0(\frac{\rho^6 h^{\frac{1}{4}}}{p_0}) \text{ as } t \to \infty, \text{ [by (2.6)]}$$

Therefore

$$\int_a^t \{y^{[3]}\bar{y} - y^{[2]}\bar{y}'\}[(1-\frac{s}{t})^2\frac{\rho^8}{p_0}]'ds$$
$$= 0(\int_a^t \{[(y^{[2]})' + p_2y' + i(p_3y'' + p_1y)]\bar{y} - y^{[2]}\bar{y}'\}\frac{\rho^6 h^{\frac{1}{4}}}{p_0}ds)$$
$$= 0(\int_0^t (y^{[2]})'\bar{y}\frac{\rho^6 h^{\frac{1}{4}}}{p_0}ds) + 0(\int_a^t \{p_2y'\bar{y} + \frac{i}{2}[p_1y$$
$$\frac{p_3}{p_0}(y^{[2]} - \frac{i}{2}p_3y')]\bar{y} - y^{[2]}\bar{y}'\}\frac{\rho^6 h^{\frac{1}{4}}}{p_0}ds).$$

Since

$$\int_a^t (y^{[2]})'\bar{y}\frac{\rho^6 h^{\frac{1}{4}}}{p_0}ds = 0(1) - \int_a^t y^{[2]}(\bar{y}\frac{\rho^6 h^{\frac{1}{4}}}{p_0})'ds$$
$$= 0(1) - \int_a^t y^{[2]}[\bar{y}'\frac{\rho^6 h^{\frac{1}{4}}}{p_0} + \bar{y}(\frac{\rho^6 h^{\frac{1}{4}}}{p_0})']ds$$

But

$$(\frac{\rho^6 h^{\frac{1}{4}}}{p_0})' = 0(\frac{\rho^4 h^{\frac{1}{2}}}{p_0}) \text{ as } t \to \infty.$$

By using Cauchy-Schwartz inequality and by Lemma (3.1) we have;

$$\begin{aligned}\int_a^t (y^{[2]})' \frac{\rho^6 h^{\frac{1}{4}}}{p_0} \bar{y} ds &= 0(W_3^{\frac{1}{2}} W_2^{\frac{1}{2}} + W_1^{\frac{1}{2}} W_3^{\frac{1}{2}}) \text{ as } t \to \infty \\ &= 0(J_1^{\frac{3}{4}}) \text{ as } t \to \infty.\end{aligned}$$

Again

$$\int_a^t p_2 \bar{y} y' \frac{\rho^6 h^{\frac{1}{4}}}{p_0} ds = 0(\int_a^t h^{\frac{1}{2}} |\bar{y}| \cdot \rho^2 h^{\frac{1}{4}} |y'|, \frac{p_2 \rho^4}{p_0 h^{\frac{1}{2}}} ds).$$

By Cauchy-Schwartz inequality and by (2.6), we get:

$$\int_a^t p_2 \bar{y} y' \frac{\rho^6 h^{\frac{1}{4}}}{p_0} ds = 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) = 0(J_1^{\frac{1}{4}}) \text{ as } t \to \infty$$

Also;

$$\begin{aligned}&\int_a^t \{p_2 y' \bar{y} + \frac{i}{2}[p_1 |y|^2 + \frac{p_3 \bar{y}}{p_0}(y^{[2]} - \frac{i p_3}{2} y')]\} \frac{\rho^6 h^{\frac{1}{4}}}{p_0} ds \\ &= \int_a \rho^2 h^{\frac{1}{4}} y' \cdot h^{\frac{1}{2}} \bar{y} \cdot \frac{p_2 \rho^4}{p_0 h^{\frac{1}{2}}} ds + \frac{i}{2} \int_a^t h |y|^2 \cdot \frac{p_1 \rho^6}{p_0 h^{\frac{3}{4}}} ds \\ &+ \frac{i}{2} \int_a^t h^{\frac{1}{2}} \bar{y} \cdot \frac{\rho^4 y^{[2]}}{p_0} \cdot \frac{p_3 \rho^2}{p_0 h^{\frac{1}{4}}} ds \\ &+ \frac{1}{4} \int_a^t h^{\frac{1}{2}} \bar{y} \cdot \rho^2 h^{\frac{1}{4}} y' \cdot \frac{p_3^2 \rho^4}{p_0^2 h^{\frac{1}{2}}} ds.\end{aligned}$$

By Cauchy-Schwartz inequality and by (2.6), we get:

$$\begin{aligned}&\int_a^t \{p_2 y' \bar{y} + \frac{i}{2}[p_1 |y|^2 + \frac{p_3 \bar{y}}{p_0}(y^{[2]} - \frac{i}{2} p_3 y')] \frac{\rho^6 h^{\frac{1}{4}}}{p_0} ds \\ &= 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}} + W_1 + W_1^{\frac{1}{2}} W_3^{\frac{1}{2}} + W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) = 0(J_1^{\frac{1}{2}}) \text{ as } t \to \infty.\end{aligned}$$

Finally

$$\begin{aligned}\int_a^t \bar{y}' y^{[2]} \frac{\rho^6 h^{\frac{1}{4}}}{p_0} ds &= 0(\int_a^t \rho^2 h^{\frac{1}{4}} \bar{y}' \cdot \frac{\rho^4 y^{[2]}}{p_0} ds \\ &= 0(J_1^{\frac{3}{4}}) \text{ as } t \to \infty\end{aligned}$$

Return to (4.2) and the left hand side can be estimated as follows:

$$\int_a^t (\lambda_h - p_4)|y|^2 \frac{\rho^8}{p_0}(1-\frac{s}{t})^2 ds = 0(\int_a^t (\lambda_h - p_4)|y|^2 \frac{\rho^8}{p_0} ds) \text{ as } t \to \infty.$$

From (2.6), we get:

$$\begin{aligned}\int_a^t (\lambda h - p_4)|y|^2 \frac{\rho^8}{p_0}(1-\frac{s}{t})^2 ds &= 0(\int_a^t h|y|^2 ds) = 0(W_1) \\ &= 0(1) \text{ as } t \to \infty\end{aligned}$$

And

$$\begin{aligned}\int_a^t p_2|y'|^2(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds &= 0(\int_a^t p_2|y'|^2 \frac{\rho^8}{p_0} ds) \\ &= 0(\int_a^t \rho^4 h^{\frac{1}{2}}|y'|^2 \cdot \frac{p_2\rho^4}{p_0 h^{\frac{1}{2}}} ds) \\ &= 0(W_2) = 0(J_1^{\frac{1}{2}}) \text{ as } t \to \infty\end{aligned}$$

Also

$$\begin{aligned}\int_a^t p_3 y''\bar{y}'(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds &= \int_a^t (y^{[2]} - \frac{i}{2}p_3 y')p_3\bar{y}'(1-\frac{s}{t})^2 \frac{\rho^8}{p_0^2} ds \\ &= 0(\int_a^t \frac{p_3\rho^8}{p_0^2}\bar{y}' y^{[2]} ds) + 0(\int_a^t \frac{p_3^2\rho^8}{p_0^2}|y'|^2 ds)\end{aligned}$$

Therefore

$$\int_a^t p_3 y''\bar{y}'(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds = 0(W_2^{\frac{1}{2}} W_3^{\frac{1}{2}}) + 0(W_2) = 0(J_1^{\frac{3}{4}}) \text{ as } t \to \infty.$$

Since

$$\begin{aligned}\int_a^t p_1 y\bar{y}'(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds &= 0(\int_a^t h^{\frac{1}{2}}|y| \cdot \rho^2 h^{\frac{1}{4}}|\bar{y}'| \frac{p_1\rho^6}{p_0 h^{\frac{3}{4}}} ds) \\ &= 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) = 0(J_1^{\frac{1}{4}}) \text{ as } t \to \infty;\end{aligned}$$

$$\int_a^t p_1 y'\bar{y}(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds = 0(J_1^{\frac{1}{4}}) \text{ as } t \to \infty$$

Finally

$$\int_a^t p_0|y''|^2(1-\frac{s}{t})^2 \frac{\rho^8}{p_0} ds = 0(t^2 J_1^{\frac{3}{4}}) \text{ as } t \to \infty$$

[By Lemma (4.2); with $F = \rho^8|y''|^2$]
Hence $J_1(\infty) < \infty$. Similarly $J_2(\infty) < \infty$.
If $Im\lambda = 0$, then $V_+ = V_-$.
If $Im\lambda \neq 0$, then $\dim V_+ + \dim V_- > 4$, and by Lemma (4.1), then $[y, z] = 1$, for $y \in V_+$ and $Z \in V_-$, i.e.

$$(y'z^{[2]} - y^{[2]}z') + (yz^{[3]} - y^{[3]}z) = 1$$

Multiplying both sides by $(1 - \frac{s}{t})\frac{\rho^6 h^{\frac{1}{4}}}{p_0}$ and integrating from $a \to t$ we get:

$$\int_a^t (y'z^{[2]} - y^{[2]}z')(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds + \int_a^t (yz^{[3]} - y^{[3]}z)(1 - \frac{s}{t})\frac{\rho^6 h^{\frac{1}{4}}}{p_0}ds$$
$$= \int_a^t (1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds \tag{4.3}$$

The first integral in the left-hand side is:

$$\begin{aligned}\int_a^t y'z^{[2]}(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds &= 0(\int_a^t h^{\frac{1}{4}}\rho^2 y' \cdot \frac{\rho^4 z^{[2]}}{p_0}ds) \text{ as } t \to \infty \\ &= 0(W_2^{\frac{1}{2}} W_3^{\frac{1}{2}}) = 0(J_1^{\frac{1}{4}} J_2^{\frac{1}{2}}) \\ &= 0([J_1 J_2]^{\frac{1}{2}}) \text{ as } t \to \infty\end{aligned}$$

Similarly

$$\begin{aligned}\int_a^t y^{[2]}z'(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds &= 0(J_1^{\frac{1}{2}} J_2^{\frac{1}{4}}) \\ &= 0([J_1 J_2]^{\frac{1}{2}}) \text{ as } t \to \infty\end{aligned}$$

The second integral can be estimated as follows:

$$\begin{aligned}\int_a^t yz^{[3]}(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds &= \int_a^t y[(z^{[2]})' + p_2 z' \\ &\quad + \frac{i}{2}\{p_1 z + \frac{p_3}{p_0}(z^{[2]} - \frac{i}{2}p_3 z')\}](1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds \\ &= \int_a^t y(z^{[2]})'(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds + \int_a^t p_2 y z'(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds \\ &\quad + \frac{i}{2}\int_a^t p_1 y z(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0} + \frac{i}{2}\int_a^t \frac{p_3}{p_0} y z^{[2]}(1 - \frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds \\ &\quad + 0(\int_a^t yz'\frac{p_3^2\rho^6 h^{\frac{1}{4}}}{p_0^2}ds)\end{aligned}$$

Since

$$\begin{aligned}\int_a^t (z^{[2]})'y(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds &= 0(\int_a^t z^{[2]}(\frac{yh^{\frac{1}{4}}\rho^6}{p_0})'ds \\ &= 0(\int_a^t \frac{\rho^4}{p_0}z^{[2]}h^{\frac{1}{4}}\rho^2 y'ds) + 0(\int_a^t \frac{\rho^4}{p_0}z^{[2]}\cdot h^{\frac{1}{2}}|y|ds) \\ &= 0(J_2^{\frac{1}{2}}J_1^{\frac{1}{4}}) + 0(J_2^{\frac{1}{2}}W_1^{\frac{1}{2}}) \\ &= 0([J_1J_2]^{\frac{1}{2}}) \text{ as } t\to\infty;\end{aligned}$$

$$\int_a^t p_2yz'(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds = 0(W_1^{\frac{1}{2}}W_2^{\frac{1}{2}}) = 0(J_2^{\frac{1}{4}}) \text{ as } t\to\infty$$

Also

$$\begin{aligned}\int_a^t p_1yz(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds &= 0(\int_a^t h^{\frac{1}{2}}y\cdot h^{\frac{1}{2}}z\frac{p_1\rho^6}{p_0h^{\frac{3}{4}}}ds) \\ &= 0(1) \text{ as } t\to\infty\end{aligned}$$

$$\begin{aligned}\int_a^t p_3yz^{[2]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0^2}ds &= 0(\int_a^t h^{\frac{1}{2}}y\cdot\frac{\rho^4z^{[2]}}{p_0}\frac{p_1\rho^2}{p_0h^{\frac{1}{4}}}ds) \\ &= 0(J_2^{\frac{1}{2}}) \text{ as } t\to\infty; and\end{aligned}$$

$$\int_a^t yz'p_3^2\frac{h^{\frac{1}{4}}\rho^6}{p_0^2}(1-\frac{s}{t})ds = 0(J_2^{\frac{1}{4}}) \text{ as } t\to\infty.$$

Hence

$$\int_a^t yz^{[3]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds = 0([J_1J_2)^{\frac{1}{2}}) \text{ as } t\to\infty.$$

Similarly

$$\int_a^t zy^{[3]}(1-\frac{s}{t})\frac{h^{\frac{3}{4}}\rho^6}{p_0}ds = 0([J_1J_2)^{\frac{1}{2}}) \text{ as } t\to\infty.$$

All these inequalities in (4.3) gives:
$\lim_{t\to\infty}\sup\int_a^t(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho^6}{p_0}ds<\infty$, and this Contrary to the condition (4.1) of the present theorem, this means that, the equation $M[y]=\lambda hy$, has at most two linearly independent solutions in $L_h^2[a,\infty)$.

Remark 1. The contents of this paper generalize the results of the author in (3) with $n=2$.

Remark 2. It is possible to compare the present results with these in (3) by choosing $P_1(x)$ and $P_3(x)$ to be mill.

References

[1] N. Dunford and J. T. Schwartz, *Linear Operator-Part II. Spectral. Theory: self-adjoint operators in Hilbert space* (Interscience, New York, 1963).

[2] W. N. Everitt, *Singular differential equations. The even-order cases*, Math. Ann. 156(1964), 9-24.

[3] Don Hinton, *Limit-Point Criteria for Differential Equations*, Can. J. Math. Vol. XXIV, No. 2 1972, pp:293-305.

[4] Don Hindon, *Limit-Point Criteria for Differential Equations II*, Can. J., Math, Vol. XXVI, No.2, 1974, pp:340-351.

[5] M. A. Naimark, *Linear Differential Operators*, Vol.II (Ungar, New-York 1968).

MATHEMATICS DEPARTMENT, FACULTY OF EDUCATION, CAIRO UNIVERSITY(FAYOUM BRANCH).