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ON VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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The existence and uniqueness of solutions of more general Volterra-Fredholm integral equations are investigated. The successive approximations method based on the general idea of T. Wazewski is the main tool.

1. Introduction

The mathematical literature on this subject provided a good information concerning the existence and uniqueness of solutions of Volterra-Fredholm integral equations by using different techniques (see [1-6]). In 1960, T. Wazewski [6] has given a general method of successive approximations which is very effective and can be applied to investigate a sufficiently wide range of problems. The purpose of this paper is to study, by using Wazeweski, the existence and uniqueness of solutions of more general Volterra-Fredholm integral equation of the form.

$$\begin{aligned} x(t) &= F[t, x(t), \int_0^t f_1(t,s,x(s))ds, \cdots, \int_0^t f_n(t,s,x(s))ds, \\ &\qquad \int_0^T g_1(t,s,x(s))ds, \cdots, \int_0^T g_n(t,s,x(s))ds] \quad 0 \le t \le T \quad (1.1) \end{aligned}$$

where $x(t)$ is an unknown function. The equation (1.1) is of more general nature and contains as special cases several types of integral equations studied by many authers as example see [1], [3] and [4].

We shall establish our main results on the existence and uniqueness of solutions of equations (1.1) by using Wazewski method.

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Our results for equation (1.1) in this general form will bring the study of a great number of integral equations under one proof and the method used in this chapter is very effective as well as versatile.

Our main hypotheses are :

Hypothesis A:

Suppose that:

I) E be a Banach space with norm $\|\cdot\|$, $I=[0;T]$, $\triangle=\{(t,s):0\leq s\leq t\leq T\}$, $f_1,\cdots,f_n,g_1,\cdots,g_n\in C[\triangle xE,E], F\in C[IxE^{2n+1},E]$ and, if $x\in C[I,E]$ and

$$\begin{aligned} z(t) &= F[t,x(t),\int_0^t f_1(t,s,x(s))ds,\cdots,\int_0^t f_n(t,s,x(s))ds, \\ &\quad \cdots,\int_0^T g_1(t,s,x(s))ds,\cdots,\int_0^T g_n(t,s,x(s))ds], \end{aligned}$$

then $Z\in C[I,E]$.

II) There exist functions $W_{11}(t,s,r),W_{12}(t,s,r),\cdots,W_{1n}(t,s,r)$, $W_{21}(t,s,r)$, $W_{22}(t,s,r),\cdots,W_{2n}(t,s,r)$ such that $W_{1i}(t,s,r)$, $W_{2i}(t,s,r)\in C[\triangle xR^+,R^+]$, $R^+=(0,\infty), i=1,\cdots,n$, which are nondecreasing in r and fulfil the conditions $W_{1i}(t,s,0)\equiv 0$, $W_{2i}(t,s,r)\equiv 0$ and $\|f_i(t,s,x)-f(t,s,\bar{x})\|\leq W_{1i}(t,s,\|x-\bar{x}\|)$, $\|g_i(t,s,x)-g(t,s,\bar{x})\|\leq W_{2i}(t,s,\|x-\bar{x}\|)$, $i=1,\cdots,n$ for $x,\bar{x}\in C[IxE]$.

III) There exists a function $H(t,r_1,r_2,r_3)$ defined for $t\in I$ and $0\leq r_1,r_2,r_3<\infty$ such that $H(t,0,0,0)\equiv 0$ and

(a) if $u\in C[I,I]$ and

$$\begin{aligned} v(t) &= H[t,u(t),\int_0^t W_{11}(t,s,u(s))ds,\cdots,\int_0^t W_{1n}(t,s,u(s))ds, \\ &\quad \int_0^T W_{21}(t,s,u(s))ds,\cdots,\int_0^T W_{2n}(t,s,u(s))ds], \end{aligned}$$

then $v\in C[I,I]$.

(b) if $u,\bar{u}\in C[I,I]$ and $u(t)\leq\bar{u}(t)$ for $t\in I$, then

$$\begin{aligned} & H[t,u(t),\int_0^t W_{11}(t,s,u(s))ds,\cdots,\int_0^t W_{1n}(t,s,u(s))ds, \\ & \quad \int_0^T W_{21}(t,s,u(s))ds,\cdots,\int_0^T W_{2n}(t,s,u(s))ds], \\ \leq\ & H[t,\bar{u}(t),\int_0^t W_{11}(t,s,\bar{u}(s))ds,\cdots,\int_0^t W_{1n}(t,s,\bar{u}(s))ds, \end{aligned}$$

$$\int_0^T W_{21}(t,s,\bar{u}(s))ds,\cdots,\int_0^T W_{2n}(t,s,\bar{u}(s))ds],$$

for $t \in I$.

(c) if $u_n \in C[I,I], u_{n+1} \leq u_n, n = 0,1,2,\cdots$, and $\lim_{n\to\infty} u_n(t) = u(t)$, then

$$\lim_{n\to\infty} H[t,u_n(t),\int_0^t W_{11}(t,s,u(s))ds,\cdots,\int_0^t W_{1n}(t,s,u_n(s))ds,$$
$$\int_0^T W_{21}(t,s,u_n(s))ds,\cdots,\int_0^T W_{2n}(t,s,u_n(s))ds],$$
$$= H(t,u(t),\int_0^t W_{11}(t,s,u(s))ds,\cdots,\int_0^t W_{1n}(t,s,u(s))ds,$$
$$\int_0^T W_{21}(t,s,u(s))ds,\cdots,\int_0^T W_{2n}(t,s,u(s))ds),$$

for $t \in I$.

IV) The inequality

$$\|F(t,x,x_i,x_j) - F(t,\bar{x},\bar{x}_i,\bar{x}_j)\|$$
$$\leq H(t,\|x-\bar{x}\|,\|x_i-\bar{x}_i\|,\|x_j-\bar{x}_j\|), \quad i,j = 1,\cdots,n$$

holds for $x,x_i,x_j,\bar{x},\bar{x}_i,\bar{x}_j \in C[I,E], t \in I$.

Hypothesis B:

Suppose that :

I) There exists a nonnegative continuous function $\bar{u} : I \to R^+$ being the solution of the inequality

$$H[t,u(t),\int_0^t W_{11}(t,s,u(s))ds,\cdots,\int_0^t W_{1n}(t,s,u(s))ds,$$
$$\int_0^T W_{21}(t,s,u(s))ds,\cdots,\int_0^T W_{2n}(t,s,u(s))ds] + h(t) \leq u(t), \quad (1.2)$$

where

$$h(t) = \sup_{t\in I} \|F(t,0,\int_0^t f_1(t,s,0)ds,\cdots,\int_0^t f_n(t,s,0)ds,$$
$$\int_0^T g_1(t,s,0)ds,\cdots,\int_0^T g_n(t,s,0)ds)\|.$$

II) In the class of functions satisfying the condition $0 \leq u(t) \leq \bar{u}(t), t \in I$, the function $u(t) \equiv 0$, $t \in I$, is the only solution of the equation

$$\begin{aligned} u(t) &= H(t, u(t), \int_0^t W_{11}(t,s,u(s))ds, \cdots, \int_0^t W_{1n}(t,s,u(s))ds, \\ &\quad \int_0^T W_{21}(t,s,u(s))ds, \cdots, \int_0^T W_{2n}(t,s,u(s))ds) \qquad (1.3) \end{aligned}$$

for $t \in I$.
In order to prove the existence of a solution of equation (1.1), we define the sequence

$$x_0(t) \equiv 0$$

$$\begin{aligned} x_{n+1}(t) &= F(t, x_n(t), \int_0^t f_1(t,s,x_n(s))ds, \cdots, \int_0^t f_n(t,s,x_n(s))ds, \\ &\quad \int_0^T g_1(t,s,x_n(s))ds, \cdots, \int_0^T g_n(t,s,x_n(s))ds) \qquad (1.4) \end{aligned}$$

for $n = 0, 1, 2, \cdots$
To prove the convergence of the sequence $\{x_n\}$ to the solution $\bar{x}$ of the equation (1.1), we define the sequence $\{u_n\}$ by the relations

$$u_0(t) = \bar{u}(t),$$

$$\begin{aligned} u_{n+1}(t) &= H(t, u_n(t), \int_0^t f_1(t,s,u_n(s))ds, \cdots, \int_0^t f_n(t,s,u_n(s))ds, \\ &\quad \int_0^T g_1(t,s,u_n(s))ds, \cdots, \int_0^T g_n(t,s,u_n(s))ds) \qquad (1.5) \end{aligned}$$

for $n = 0, 1, 2, \cdots$, where the function $\bar{u}(t)$ is from hypothesis B.

Now we establish the following basic lemma needed in our subsequent discussion.

Lemma 1.1. *If condition (III) of hypothesis A and hypothesis B are satisfied, then*

$$0 \leq u_{n+1}(t) \leq u_n(t) \leq \bar{u}(t), \quad t \in I, n = 0, 1, 2, \cdots \qquad (1.6)$$

$$\lim_{n \to \infty} u_n(t) = 0, \quad t \in I,$$

and the convergence is uniform in each bounded set.

Proof. From (1.5) and (1.2) we have

$$\begin{aligned}
u_1(t) &= H(t, u_0(t), \int_0^t f_1(t,s,u_0(s))ds, \cdots, \int_0^t f_n(t,s,u_0(s))ds, \\
&\qquad \int_0^T g_1(t,s,u_0(s))ds, \cdots, \int_0^T g_n(t,s,u_0(s))ds) \\
&\leq H(t, \bar{u}(t), \int_0^t f_1(t,s,\bar{u}(s))ds, \cdots, \int_0^t f_n(t,s,\bar{u}(s))ds, \\
&\qquad \int_0^T g_1(t,s,\bar{u}(s))ds, \cdots, \int_0^T g_n(t,s,\bar{u}(s))ds) + h(t) \\
\bar{u}(t) &= u_0(t)
\end{aligned}$$

for $t \in I$. Further we obtain (1.6) by induction. But (1.6) implies the convergence of the sequence $\{u_n(t)\}$ to some non-negative function $\phi(t)$ for $t \in I$. By Lebesgue's theorem and the continuity of H it follows that the function $\phi(t)$ satisfies equation (1.3). Now from hypothesis B, we have $\phi(t) \equiv 0$, $t \in I$. The uniform convergence of the sequence $\{u_n\}$ in I follows from Dini's theorem. Thus the proof of Lemma 1.1 is complete.

2. Main Results

We establish our main results on the existence and uniqueness of the solutions of equation (1.1).

Theorem 2.1. *If hypotheses A and B are satisfied, then there exists a continuous solution $\bar{x}$ of equation (1.1). The sequence $\{x_n\}$ defined by (1.4) converges uniformly on I to $\bar{x}$, and the following estimates*

$$\|\bar{x}(t) - x_n(t)\| \leq u_n(t), \quad t \in I, n = 0, 1, 2, \cdots \tag{2.1}$$

and

$$\|\bar{x}(t)\| \leq \bar{u}(t), \quad t \in I \tag{2.2}$$

hold. The solution $\bar{x}$ of equation (1.1) is unique in the class of functions satisfying the condition (2.2).

Proof. We first prove that the sequence $\{x_n(t)\}$, $t \in I$, fulfils the condition

$$\|x_n(t)\| < \bar{u}(t), \quad t \in I, \quad n = 0, 1, 2, \cdots \tag{2.3}$$

evidently, we see that

$$\|x_0(t)\| \equiv 0 \leq \bar{u}(t), \quad t \in I.$$

Further, if we suppose that the inequality (2.3) is true for $n \geq 0$, then

$$\begin{aligned}
\|x_{n+1}(t)\| = \|&F(t, x_n(t), \int_0^t f_1(t,s,x_n(s))ds, \cdots, \int_0^t f_n(t,s,x_n(s))ds,\\
&\int_0^T g_1(t,s,x_n(s))ds, \cdots, \int_0^T g_n(t,s,x_n(s))ds)\\
&-F(t, 0, \int_0^t f_1(t,s,0)ds, \cdots, \int_0^t f_n(t,s,0)ds,\\
&\int_0^T g_1(t,s,0)ds, \cdots, \int_0^T g_n(t,s,0)ds)\\
&+F(t, 0, \int_0^t f_1(t,s,0)ds, \cdots, \int_0^t f_n(t,s,0)ds,\\
&\int_0^T g_1(t,s,0)ds, \cdots, \int_0^T g_n(t,s,0)ds)\|\\
\leq H(t, &\|x_n(t)\|, \int_0^t W_{11}(t,s,\|x_n(s)\|)ds, \cdots, \int_0^t W_{1n}(t,s,\|x_n(s)\|)ds,\\
&\int_0^T W_{21}(t,s,\|x_n(s)\|)ds, \cdots, \int_0^T W_{2n}(t,s,\|x_n(s)\|)ds) + h(t)\\
\leq H(t, &\bar{u}(t), \int_0^t W_{11}(t,s,\bar{u}(s))ds, \cdots, \int_0^t W_{1n}(t,s,\bar{u}(s))ds\\
&\int_0^T W_{21}(t,s,\bar{u}(s))ds, \cdots, \int_0^T W_{2n}(t,s,\bar{u}(s))ds) + h(t)\\
\leq \bar{u}&(t)
\end{aligned}$$

for $t \in I$. Now we obtain (2.3) by induction.
Next we prove that

$$\|x_{n+1}(t) - x_n(t)\| \leq u_n(t), \quad t \in I, \quad \begin{array}{rcl} n &=& 0, 1, 2, \cdots \\ r &=& 0, 1, 2, \cdots \end{array} \tag{2.4}$$

By (2.3) we have

$$\|x_r(t) - x_0(t)\| = \|x_r(t)\| \leq \bar{u}(t) = u_0(t), t \in I, r = 0, 1, 2, \cdots$$

Suppose that (2.4) is true for n, $r \geq 0$ then

$$\begin{aligned}
\|x_{n+r+1}(t) - x_{n+1}(t)\| = \|&F(t, x_{n+r}(t), \int_0^t f_1(t,s,x_{n+r}(s))ds, \cdots,\\
&\int_0^t f_n(t,s,x_{n+r}(s))ds,\\
&\int_0^T g_1(t,s,x_{n+r}(s))ds, \cdots, \int_0^T g_n(t,s,x_{n+r}(s))ds)
\end{aligned}$$

$$-\|F[(t,x_n(t),\int_0^t f_1(t,s,x_n(s))ds,\cdots,\int_0^t f_n(t,s,x_n(s))ds,$$
$$\int_0^T g_1(t,s,x_n(s))ds,\cdots,\int_0^T g_n(t,s,x_n(s))ds]\|$$
$$\le H(t,\|x_{n+r}(t)-x_n(t)\|,\int_0^t W_{11}(t,s,\|x_n(s)\|)ds,\cdots,$$
$$\int_0^t W_{1n}(t,s,\|x_{n+r}(s)-x_n(s)\|)ds,\int_0^T W_{21}(t,s,\|x_{n+r}(s)-x_n(s)\|)ds,\cdots,$$
$$\int_0^T W_{2n}(t,s,\|x_{n+r}(s)-x_n(s)\|)ds)$$
$$\le H(t,u_n(t),\int_0^t W_{11}(t,s,u_n(s))ds,\cdots,\int_0^t W_{1n}(t,s,u_n(s))ds,$$
$$\int_0^T W_{21}(t,s,u_n(s))ds,\cdots,\int_0^T W_{2n}(t,s,u_n(s))ds)$$
$$=u_{n+1}(t)$$

for $t\in I$. Now we obtain (2.4) by induction. Because of lemma 1.1, $\lim_{n\to\infty}u_n(t)=0$ in I, we have from (2.4) $x_n\to\bar{x}$ in I. The continuity of $\bar{x}$ follows from the uniform convergence of the sequence $\{x_n\}$ and the continuity of all functions x_n. If $r\to\infty$, then (2.4) gives estimation (2.1). Estimation (2.2) is implied by (2.3). It is obvious that $\bar{x}$ is a solution of equation (1.1).

To prove that the solution $\bar{x}$ is a unique solution of equation (1.1) in the class of functions satisfying the condition (2.2). Let us suppose that there exists another solution $\hat{x}$ defined in I and such that $\bar{x}(t)\not\equiv\hat{x}(t)$ for $t\in I$ and $\|\hat{x}(t)\|\le\bar{u}(t)$ for $t\in I$. From (2.1) we get $\|\hat{x}(t)-x_n(t)\|u_n(t)$, $t\in I$, $n=0,1,2,\cdots$ and it follows that $\bar{x}(t)=\hat{x}(t)$ for $t\in I$. This contradiction proves the uniqueness of $\bar{x}$ in the class of functions satisfying relation (2.2). This completes the proof of the theorem.

We next establish a theorem which give conditions under which equation (1.1) has at most one solution. These conditions do not quarantee existence.

Theorem 2.2. *If hypothesis A is satisfied and the function $m(t)\equiv 0$, $t\in I$ is the only nonnegative continuous solution of the inequality*

$$m(t) \;\le\; H((t,m(t),\int_0^t W_{11}(t,s,m(s))ds,\cdots,\int_0^t W_{1n}(t,s,m(s))ds$$
$$\int_0^T W_{21}(t,s,m(s))ds,\cdots,\int_0^T W_{2n}(t,s,m(s))ds),$$

$$0 \leq t \leq T, \tag{2.5}$$

then equation (1.1) has at most one solution.

Proof. Let us suppose that there exist two solutions $\bar{x}$ and $\hat{x}$ of equation (1.1) such that $\bar{x}(t) \not\equiv \hat{x}(t)$, $t \in I$. Put $m(t) = \|\bar{x}(t) - \hat{x}(t)\|, t \in I$, then

$$\begin{aligned}
m(t) &= \|F(t, \bar{x}(t), \int_0^t f_1(t,s,\bar{x}(s))ds, \cdots, \int_0^t f_n(t,s,\bar{x}(s))ds, \\
&\quad \int_0^T g_1(t,s,\bar{x}(s))ds, \cdots, \int_0^T g_n(t,s,\bar{x}(s))ds) \\
&\quad -F(t, \hat{x}(t), \int_0^t f_1(t,s,\hat{x}(s))ds, \cdots, \int_0^t f_n(t,s,\hat{x}(s))ds, \\
&\quad \int_0^T g_1(t,s,\hat{x}(s))ds, \cdots, \int_0^T g_n(t,s,\hat{x}(s))ds)\| \\
&\leq H(t, \|\bar{x}(t) - \hat{x}(t)\|, \int_0^t W_{11}(t,s,\|\bar{x}(s) - \hat{x}(s)\|)ds, \cdots, \\
&\quad \int_0^t W_{1n}(t,s,\|\bar{x}(s) - \hat{x}(s)\|)dsds, \\
&\quad \int_0^T W_{21}(t,s,\|\bar{x}(s) - \hat{x}(s)\|)ds, \cdots, \int_0^T W_{2n}(t,s,\|\bar{x}(s) - \hat{x}(s)\|)ds) \\
&= H(t, m(t), \int_0^t W_{11}(t,s,m(s))ds, \cdots, \int_0^t W_{1n}(t,s,m(s))ds, \\
&\quad \int_0^T W_{21}(t,s,m(s))ds, \cdots, \int_0^T W_{2n}(t,s,m(s))ds)
\end{aligned}$$

and by (2.5) we conclude that $m(t) \equiv 0$ for $t \in I$, i.e. $\bar{x}(t) = \hat{x}(t), t \in I$. This contradiction proves our theorem.

Remarks.

(1) Asirov and Mamedov [1] and Mamedov and Musaev [4] have studied a special case of equation (1.1) of the form

$$x(t) = F(t, \int_0^t f(t,s,x(s))ds, \int_0^T g(t,s,x(s))ds), \quad 0 \leq t \leq T$$

(2) Equation (1.1) in turn can be carridered as a further generalization of the nonlinear volterra integral equation studied by Grossman [3].

Key words and phrases. Volterra-Fredholm integral equations, existence and uniqueness of solutions.

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