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DISTORTION THEOREMS FOR ALPHA-STARLIKE FUNCTIONS

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The object of the present paper is to prove some interesting distortion theorems for alpha-starlike functions

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z)$ belonging to A is said to be starlike of order α if and only if

$$Re\{\frac{zf'(x)}{f(z)}\} > \alpha \tag{1.2}$$

for some $\alpha (0 \leq \alpha < 1)$, and for all $z \in U$. We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in the unit disk U. Note that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$ for $0 \leq \alpha < 1$.

A function $f(z)$ belonging to A is said to be convex of order α if and only if

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha \tag{1.3}$$

for some $\alpha (0 \leq \alpha < 1)$, and for all $z \in U$. Also we denote by $K(\alpha)$ the subclass of A consisting of such functions. We note that $K(\alpha) \subseteq K(0) \equiv K$ for $0 \leq \alpha < 1$, and that $K(\alpha) \subset S^*(\alpha)$ for $0 \leq \alpha < 1$.

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Let α be real and suppose that $f(z)$ belongs to A. If $f(z)$ satisfies

$$(1.4) \qquad Re\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(1+\frac{zf''(z)}{f'(z)})\}>0$$

for some $\alpha(-\infty \leq \alpha \leq \infty)$, and for all $z \in U$, then $f(z)$ is said to be α-starlike in the unit disk U. We let the class of functions which are α-starlike in the unit disk U be denoted by M_α.

The class M_α was first introduced by Mocanu [4], and was studied by Mocanu and Reade [5], Miller, Mocanu and Reade [3], Miller [2], and Sakaguchi and Fukui [7].

2. Distortion Theorems

In order to prove some distortion theorems for functions belonging to M_α, we have to recall here the following lemmas.

Lemma 1 ([7]). *If $f(z) \in M_\alpha$ with $0 \leq \alpha < 1$, then $f(z) \in S^*$. If $f(z) \in M_\alpha$ with $\alpha \geq 1$, then $f(z) \in K$.*

Lemma 2 ([1]). *If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where*

$$(2.1) \qquad \beta(\alpha)=\begin{cases}\frac{2\alpha-1}{2(1-2^{1-2\alpha})} & (\alpha \neq \frac{1}{2})\\ \frac{1}{2\log 2} & (\alpha=\frac{1}{2}).\end{cases}$$

Lemma 3. *If $f(z) \in M_\alpha$ with $\alpha \geq 1$, then $f(z) \in K(\frac{\alpha-1}{2\alpha})$.*

Proof. By using Lemma 1 and Lemma 2, we note that if $f(z) \in M_\alpha$ with $\alpha \geq 1$, then $f(z) \in K \subset S^*(\frac{1}{2})$, that is, that

$$(2.2) \qquad Re\{\frac{zf'(z)}{f(z)}\}>\frac{1}{2} \quad (z \in U).$$

Therefore, from (1.4), we have

$$(2.3) \qquad Re\{1+\frac{zf''(z)}{f'(z)}\}>(\frac{\alpha-1}{\alpha})Re\{\frac{zf'(z)}{f(z)}\}>\frac{\alpha-1}{2\alpha}$$

which implies $f(z) \in K(\frac{\alpha-1}{2\alpha})$.

Lemma 4 ([2]). *If $f(z) \in M_\alpha$ with $\alpha > 0$, then for $|z|=r<1$ we have*

$$(2.4) \qquad -K(\alpha,-r) \leq |f(z)| \leq K(\alpha,r),$$

where

$$K(\alpha,r)=\Big\{\frac{1}{\alpha}\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}d\rho\Big\}^{\alpha}. \tag{2.5}$$

Equality holds in both cases for the function

$$f_\theta(\alpha,z)=\Big\{\frac{1}{\alpha}\int_0^z \zeta^{1/\alpha-1}(1-\zeta e^{i\theta})^{-2/\alpha}d\zeta\Big\}^{\alpha}. \tag{2.6}$$

Lemma 5 ([6]). *If $f(z)\in S^*(\alpha)$ with $0\le\alpha<1$, then for $|z|=r<1$ we have*

$$\Big|\frac{zf'(z)}{f(z)}\Big|\le\frac{r\log\{\frac{(1+r)^{2(1-\alpha)}}{r}|f(z)|\}}{(1-r)\log(\frac{1+r}{1-r})}+1. \tag{2.7}$$

Equality in (2.7) holds true for the function $f(z)=z/(1-z)^{2(1-\alpha)}$ with $z=r$.

Lemma 6 ([8]). *$f(z)\in S^*(\alpha)$ with $0\le\alpha<1$, then for $|z|=r<1$ we have*

$$Re\{\frac{zf'(z)}{f(z)}\}\ge\alpha+(1-\alpha)\frac{1-r^2}{r^{1/(1-\alpha)}}|f(z)|^{1/(1-\alpha)} \tag{2.8}$$

and

$$Re\{\frac{zf'(z)}{f(z)}\}\le\frac{1+(1-2\alpha)r}{1-r}+\frac{2r\log\{\frac{(1-r)^{2(1-\alpha)}}{r}|f(z)|\}}{(1-r^2)\log(\frac{1+r}{1-r})}. \tag{2.9}$$

For the functions $f(z)$ belonging to M_α, Miller [2] gave the following conjecture.

Conjecture. *If $f(z)\in M_\alpha$ with $\alpha>0$, then*

$$(\partial/\partial r)K(\alpha,-r)\le|f'(z)|\le(\partial/\partial r)K(\alpha,r), \tag{2.10}$$

where $K(\alpha,r)$ is given by (2.5).

Furthermore, Miller [2] proved the above conjecture for $\alpha\ge 1$. Now, we prove

Theorem 1. *If $f(z)\in M_\alpha$ with $0\le\alpha<1$, then*

$$|f'(z)|\le\frac{K(\alpha,r)}{r}\Big\{\frac{r\log\{\frac{(1+r)^2}{r}K(\alpha,r)\}}{(1-r)\log(\frac{1+r}{1-r})}+1\Big\} \tag{2.11}$$

for $|z|=r<1$, where $K(\alpha,r)$ is given by (2.5). Equality in (2.11) holds true for the function $f(z)=z/(1-z)^{2(1-\alpha)}$ with $z=r$.

Proof. Note that Lemma 1 gives $f(z) \in S^*$ for functions $f(z)$ belonging to M_α with $0 \leq \alpha < 1$. Applying Lemma 5 when $\alpha = 0$ and Lemma 4, we can show the inequality (2.11).

Combining Lemma 2, Lemma 3, Lemma 4, and Lemma 5, we have

Theorem 2. *If* $f(z) \in M_\alpha$ *with* $\alpha \geq 1$, *then*

$$(2.12) \qquad |f'(z)| \leq \frac{K(\alpha, r)}{r}\left\{\frac{r \log\{\frac{(1+r)^{2(1-\gamma(\alpha))}}{r} K(\alpha, r)\}}{(1-r)\log(\frac{1+r}{1-r})} + 1\right\}$$

for $|z| = r < 1$, *where* $K(\alpha, r)$ *is given by (2.5) and*

$$(2.13) \qquad \gamma(\alpha) = \begin{cases} \frac{(\alpha-1)/\alpha-1}{2(1-2^{1-(\alpha-1)/\alpha})} & (1 \leq \alpha < \infty) \\ \frac{1}{2\log 2} & (\alpha = \infty). \end{cases}$$

Proof. In view of Lemma 2 and Lemma 3, we have $f(z) \in S^*(\gamma(\alpha))$, where $\gamma(\alpha)$ is defined by (2.13). Therefore, Lemma 4 and Lemma 5 imply the inequality (2.12).

With the aid of Lemma 6, we have

Theorem 3. *If* $f(z) \in M_\alpha$ *with* $0 \leq \alpha < 1$, *then*

$$(2.14) \qquad Re\{\frac{zf'(z)}{f(z)}\} \geq \frac{r^2-1}{r} K(\alpha, -r)$$

and

$$(2.15) \qquad Re\{\frac{zf'(z)}{f(z)}\} \leq \frac{1+r}{1-r} + \frac{2r\log\{\frac{(1-r)^2}{r}K(\alpha, r)\}}{(1-r^2)\log(\frac{1+r}{1-r})}$$

for $|z| = r < 1$, *where* $K(\alpha, r)$ *is given by (2.5).*

Proof. Applying Lemma 1, Lemma 4, and Lemma 6 when $\alpha = 0$, we can easily show the inequalities (2.14) and (2.15).

Finally, we prove

Theorem 4. *If* $f(z) \in M_\alpha$ *with* $\alpha \geq 1$, *then*

$$(2.16) Re\{\frac{zf'(z)}{f(z)}\} \geq \gamma(\alpha) + (1-\gamma(\alpha))\frac{1-r^2}{r^{1/(1-\gamma(\alpha))}}(-K(\alpha, -r))^{1/(1-\gamma(\alpha))})$$

and

$$(2.17) \quad \{\frac{zf'(z)}{f(z)}\} \leq \frac{1+(1-2\gamma(\alpha))r}{1-r} + \frac{2r\log\{\frac{(1-r)^{2(1-\gamma(\alpha))}}{r}K(\alpha, r)\}}{(1-r^2)\log(\frac{1+r}{1-r})}$$

for $|z| = r < 1$, *where* $K(\alpha, r)$ *and* $\gamma(\alpha)$ *are defined by (2.5) and (2.13), respectively.*

Proof. Combining Lemma 2, Lemma 3, Lemma 4, and Lemma 6, we can prove the inequalities (2.16) and (2.17).

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