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AN INVENTORY MODEL AND ITS OPTIMIZATION

Eui Yong Lee and Won J. Park

An inventory model with constant demand of rate $\mu(\mu > 0)$ is considered. The inventory is replenished up to β by a deliveryman who arrives according to a Poisson process of rate λ, only if the stock does not exceed a threshold $\alpha(0 \leq \alpha \leq \beta)$. The distribution function of $X(t)$, the stock at time t, is deduced from a partial differential equation, two interesting characteristics, the first passage time to state 0 and the probability that the stock exceeds a certain level during a given interval, are considered, the stationary distribution is obtained more explicitly, and an optimal policy with respect to the threshold α is studied.

1. Introduction

In this paper, an inventory model is introduced. Consider an inventory whose stock is initially β, thereafter decreases linearly at rate μ, $\mu > 0$, and remains at 0 if the inventory becomes empty. The inventory is replenished by a deliveryman who arrives at the inventory according to a Poisson process of rate λ. If the level of the inventory exceeds a threshold α, $0 \leq \alpha \leq \beta$, he does nothing, otherwise he instantaneously increases the level of the inventory up to β.

Baxter and Lee [1] introduced a similar inventory model where the size of a delivery is a random variable Y such that $Y \geq \alpha$ almost surely. In the paper, they derived a Laplace-Stieltjes transform of the distribution function of the level of the inventory at time t and considered the stationary case where the distribution function does not depend on time t.

Since the inventory is replenished up to β in our model rather than by a random amount (β may be considered as the capacity of the inventory),

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the points where the restockings occur form a renewal process, and this fact enables us to obtain the distribution function of the level of the inventory at time t directly and to study the stationary case more explicitly. We further consider two interesting characteristics of the model, the first passage time to state 0 and the probability that the stock exceeds a certain level during a given interval. We also show that there exists a unique optimal policy with respect to the threshold α, after assigning costs to the inventory.

2. The Distribution Function

Let $X(t)$ be the level of the inventory at time t and $F(x,t)$ be the distribution function of $X(t)$. We can have the following three mutually exclusive events during the small interval $(t, t+\delta t)$:

(a) The deliveryman does not come, then

$$X(t+\delta t) = \begin{cases} X(t) - \mu\delta t & \text{almost surely if } X(t) > \mu\delta t \\ 0 & \text{almost surely if } X(t) \le \mu\delta t. \end{cases}$$

(b) The deliveryman comes but does nothing since $X(t) > \alpha$, then

$$X(t+\delta t) = X(t) - \mu\delta t \quad \text{almost surely.}$$

(c) The deliveryman comes and makes a delivery since $X(t) \le \alpha$ then

$$X(t+\delta t) = \beta - \mu\delta t \quad \text{almost surely.}$$

Thus, for $0 \le x < \beta$,

$$\begin{aligned} F(x,t+\delta t) &= (1-\lambda\delta t)F(x+\mu\delta t,t) + \lambda\delta t P\{X(t) \le x+\mu\delta t, \\ &\qquad X(t) > \alpha\} + \lambda\delta t I\{x \ge \beta - \mu\delta t\}F(\alpha,t) + o(\delta t), \end{aligned}$$

where I_A denotes the indicator of event A. Now

$$F(x+\mu\delta t,t) = F(x,t) + \mu\delta t\frac{\partial}{\partial x}F(x,t) + o(\delta t)$$

on performing a Taylor series expansion, assuming that $\frac{\partial}{\partial x}F(x,t)$ exists. Substituting this expression into the above equation, subtracting $F(x,t)$ from each side of the equation, dividing by δt, and letting $\delta t \to 0$, we have the following partial differential equation:

$$\frac{\partial}{\partial t}F(x,t) = \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(x\wedge\alpha,t), \quad \text{for} \quad 0 \le x < \beta. \tag{2.1}$$

Since the level of the inventory cannot exceed β, $F(\beta, t) = 1$ for $t > 0$.

Before we solve the equation (2.1), we first derive a formula for $F(\alpha, t)$, which can be used as a boundary condition.

Lemma 2.1. *If we ignore the first passage time to* α, *i.e.* $\frac{\beta-\alpha}{\mu}$, *then*

$$F(\alpha,t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} h(u)du,$$

where $h(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$ *and* $g(t) = \lambda e^{-\lambda(t-\frac{\beta-\alpha}{u})}$.

Proof. Notice that the points where the stock of the inventory reaches α from an embedded renewal process. Let T^* be the time between successive renewals. Then

$$T^* = T + \frac{\beta-\alpha}{\mu},$$

where T is an exponential random variable with parameter λ. The probability density function of T^*, $g(t)$ say, is given by

$$g(t) = \lambda e^{-\lambda(t-\frac{\beta-\alpha}{\mu})}, \quad \text{for} \quad t > \frac{\beta-\alpha}{\mu}.$$

Let $h(t)$ denote the renewal density function of the embedded renewal process, that is, $h(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$, where the superscript denotes n-fold recursive convolution.

Now, notice that $F(\alpha, t) = 1$ if the deliveryman has not arrived until time t or if there is a renewal in the embedded renewal process at $u \in (0, t]$ and the deliveryman does not arrive in the interval $[u, t]$. Hence

$$F(\alpha,t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} h(u)du.$$

Now, the equation (2.1) can be divided into the following two equations:

$$\frac{\partial}{\partial t}F(x,t) = \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(x,t), \quad \text{for} \quad 0 \le x < \alpha,$$

and

$$\frac{\partial}{\partial t}F(x,t) = \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(\alpha,t), \quad \text{for} \quad \alpha \le x < \beta.$$

Applying $F(\alpha, t)$ obtained in Lemma 2.1 to both equations as a boundary condition and solving the partial differential equations for $F(x, t)$ by an argument similar to that of Colton [2, p.6-11], we see that

$$F(x,t) = F(\alpha, t+\frac{x-\alpha}{\mu})e^{\lambda(x-\alpha)/\mu}, \quad \text{for} \quad 0 \le x < \alpha, \quad \text{and}$$

$$F(x,t) = F(\alpha, t+\frac{x-\alpha}{\mu}) + \frac{\lambda}{\mu}\int_\alpha^x F(\alpha, t+\frac{x-u}{\mu})du, \quad \text{for} \quad \alpha \le x < \beta,$$

where $F(\alpha,t)$ is given in Lemma 2.1.

3. The First Passage Time to State 0

Define $T_0 = \inf\{t|X(t) = 0\}$, the first passage time to state 0. Let $Y_1, Y_2, \cdots, Y_N$ be the sequence of the amounts of the deliveries made by the deliveryman, before the stock reaches state 0, then

$$Y_i \stackrel{\mathcal{D}}{=} \beta - \alpha + \mu T, \quad i = 1, 2, \cdots, N,$$

under the condition that T, the exponential random variable with parameter λ, is less than $\frac{\alpha}{\mu}$, and so the distribution function of Y_i, $D(y)$ say, is given by

$$\begin{aligned} D(y) &= P(\beta - \alpha + \mu T \le y | T < \frac{\alpha}{\mu}) \\ &= \begin{cases} 0, & \text{for } y \le \beta - \alpha \\ \frac{1-e^{-\lambda(y-\beta+\alpha)/\mu}}{1-e^{-\alpha\lambda/\mu}}, & \text{for } \beta - \alpha < y \le \beta \\ 1, & \text{for } y > \beta. \end{cases} \end{aligned}$$

Further,

$$P(N = n) = e^{-\alpha\lambda/\mu}(1 - e^{-\alpha\lambda/\mu})^n, \quad n = 0, 1, 2, \cdots.$$

Now, observed that T_0 satisfies the following relation :

$$T_0 \stackrel{\mathcal{D}}{=} \frac{1}{\mu}(\beta + \sum_{i=0}^{N} Y_i)$$

and hence the distribution function of T_0, $L(t)$ say, is given by

$$L(t) = \sum_{n=0}^{\infty} D^{(n)}(\mu t - \beta)e^{-\alpha\lambda/\mu}(1 - e^{-\alpha\lambda/\mu})^n,$$

where $D^{(n)}$ is the n-fold recursive Stieltjes convolution of D, $D^{(0)}$ being the Heaviside function. It can be also shown that

$$E(T_0) = \frac{\beta - \alpha}{\mu}e^{\alpha\lambda/\mu} + \frac{1}{\lambda}(e^{\alpha\lambda/\mu} - 1).$$

4. The Probability That the Stock Exceeds a Given Level

We now derive an expression for $\pi_x(t_1, t_2) = P\{X(t) > x$, for all $t \in [t_1, t_2]\}$. Since the result is trivial if $x \geq \alpha$, we consider only the case when $x < \alpha$. Observe that $X(t) > x$ for all $t \in [t_1, t_2]$ if and only if $X(t_1) > x$ and the first passage time from $X(t_1)$ to x is greater than $t_2 - t_1$. Let S_{y-x} denote the first passage time from state y to state x, then

$$\begin{aligned}\pi_x(t_1, t_2) &= P\{X(t_1) > x, S_{X(t_1)} - x > t_2 - t_1\} \\ &= \int_x^\beta P\{S_{y-x} > t_2 - t_1 | X(t_1) = y\} dF(y, t_1)\end{aligned}$$

by conditioning on $X(t_1)$. Let $L_x^y(t)$ denote the distribution function of S_{y-x}. By an argument similar to that of the previous section, it can be shown that

$$\begin{aligned}L_x^y(t) &= D^{(0)}(\mu t + x - y)e^{-\lambda((\alpha \wedge y)-x)/\mu} \\ &\quad + \sum_{n=1}^{\infty} D_x^{(n)}(\mu t + x - y)(1 - e^{-\lambda((\alpha \wedge y)-x)/\mu}) \\ &\quad e^{-\lambda(\alpha-x)/\mu}(1 - e^{-\lambda(\alpha-x)/\mu})^{n-1},\end{aligned}$$

where

$$D_x(y) = \begin{cases} 0, & \text{for } y < \beta - \alpha \\ \frac{1-e^{-\lambda(y-\beta+\alpha)/\mu}}{1-e^{-\lambda(\alpha-x)/\mu}}, & \text{for } \beta - \alpha < y \leq \beta - x \\ 1, & \text{for } y > \beta - x. \end{cases}$$

Summarizing the foregoing, we see that

$$\pi_x(t_1, t_2) = \int_x^\beta L_x^y(t_2 - t_1) dF(y, t_1).$$

5. The Stationary Case

In this section, we consider the case where the distribution function of $X(t)$ does not depend on time t, that is, $\partial F(x,t)/\partial t = 0$. Notice that this stationary distribution is the same as the equilibrium distribution $F(x) = \lim_{t\to\infty} F(x,t)$ (c.f. Baxter and Lee [1]).

From the equation (2.1), it follows that

$$\mu\frac{d}{dx}F(x) - \lambda F(x) = 0, \quad \text{for} \quad 0 \le x < \alpha \tag{5.1}$$

$$\mu\frac{d}{dx}F(x) - \lambda F(\alpha) = 0, \quad \text{for} \quad \alpha \le x < \beta \tag{5.2}$$

Applying the key renewal theorem to $F(\alpha, t)$ obtained in Lemma 2.1, we see that

$$F(\alpha) = \frac{\mu}{\mu + \lambda(\beta - \alpha)}. \tag{5.3}$$

Hence, solving the equations (5.1) and (5.2) with the boundary condition given by the equation (5.3), we obtain

$$\begin{aligned} F(x) &= \frac{\mu e^{\lambda(x-\alpha)/\mu}}{\mu + \lambda(\beta - \alpha)}, \quad \text{for} \quad 0 \le x < \alpha, \quad \text{and} \\ F(x) &= \frac{\mu - \alpha\lambda + \lambda x}{\mu + \lambda(\beta - \alpha)}, \quad \text{for} \quad \alpha \le x < \beta. \end{aligned}$$

From the above stationary distribution, it can be also shown that the average level of the inventory over an infinite horizon is given by

$$\frac{1}{\mu + \lambda(\beta - \alpha)}[\alpha\mu + \frac{\lambda(\beta^2 - \alpha^2)}{2} - \mu^2(1 - e^{-\alpha\lambda/\mu})/\lambda].$$

6. The Optimal Policy with Respect to α

In this section, we show that there exists a unique α which minimizes the average cost per unit time over an infinite horizon, after assigning costs to the inventory, the cost per unit time of the inventory being empty, C_1 say, and the cost of keeping a unit per unit time, C_2 say.

To calculate $C(\alpha)$, the average cost per unit time over an infinite horizon, we define as a cycle the interval between two successive points where the inventory is replenished up to β. Notice again that the sequence of such points forms an embedded renewal process. The duration of a generic interval is denoted T^*. It can be shown that the total cost during a cycle is given by

$$\begin{aligned} &C_2 \int_0^{(\beta-\alpha)/\mu+T} (\beta - \mu x)dx, \quad \text{if} \quad T < \alpha/\mu, \\ &C_1(T - \alpha/\mu) + C_2\frac{\beta^2}{2\mu}, \qquad \textit{otherwise}, \end{aligned}$$

where T is an exponential random variable with parameter λ. Hence, the expected total cost in a cycle can be obtained by conditioning on T,

$$\begin{aligned}\hat{C}(\alpha) &= C_1\int_{\alpha/\mu}^{\infty}(t-\alpha/\mu)\lambda e^{-\lambda t}dt \\ &\quad + C_2\int_0^{\alpha/\mu}\int_0^{(\beta-\alpha)/\mu+t}(\beta-\mu x)dx\lambda e^{-\lambda t}dt + C_2\int_{\alpha/\mu}^{\beta}\frac{\beta^2}{2\mu}\lambda e^{-\lambda t}dt \\ &= C_1 e^{-\alpha\lambda/\mu}/\lambda + C_2\left(\frac{\beta^2}{2\mu}-\frac{\alpha^2}{2\mu}+\frac{\alpha}{\lambda}+\frac{\mu}{\lambda^2}e^{-\alpha\lambda/\mu}-\frac{\mu}{\lambda^2}\right).\end{aligned}$$

Since $C(\alpha) = \hat{C}(\alpha)/E(T^*)$ and $E(T^*) = ((\beta-\alpha)/\mu + 1/\lambda)$, it follows that

$$C(\alpha) = \frac{1}{(\beta-\alpha)\lambda+\mu}[C_1\mu e^{-\alpha\lambda/\mu}+C_2(\beta^2\lambda^2-\alpha^2\lambda^2+2\alpha\mu\lambda+2\mu^2e^{-\alpha\lambda/\mu}-2\mu^2)/2\lambda].$$

Theorem 6.1. *If $C_1 \leq C_2\beta/2$, then $C(\alpha)$ is minimized at $\alpha = 0$, if $C_1 \geq C_2\mu(e^{\lambda\beta/\mu}-1)/\lambda$, then $C(\alpha)$ is minimized at $\alpha = \beta$, otherwise, there exists a unique α^*, $0 < \alpha^* < \beta$, which minimized $C(\alpha)$.*

Proof. First, $C'(\alpha)$ is given by

$$\begin{aligned}C'(\alpha) &= \frac{-\lambda(\beta-\alpha)}{((\beta-\alpha)\lambda+\mu)^2}[(C_1\lambda+\mu C_2)e^{-\alpha\lambda/\mu}+C_2\alpha\lambda/2-C_2\beta\lambda/2-C_2\mu] \\ &= \frac{-\lambda(\beta-\alpha)}{((\beta-\alpha)\lambda+\mu)^2}[A_1(\alpha)-A_2(\alpha)],\end{aligned}$$

$$\begin{aligned}\text{where} A_1(\alpha) &= (C_1\lambda+C_2\mu)e^{-\alpha\lambda/\mu} \quad \text{and} \\ A_2(\alpha) &= -C_2\alpha\lambda/2+C_2\beta\lambda/2+C_2\mu.\end{aligned}$$

Notice that $A_1(\alpha)$ is an exponential function of α and $A_2(\alpha)$ is a linear function of α. There are three cases to consider :

(i) when $C_1 \leq C_2\beta/2$.

Since $A_1(0) \leq A_2(0)$ and $A_1(\beta) \leq A_2(\beta)$, $A_1(\alpha) \leq A_2(\alpha)$, for all $0 \leq \alpha \leq \beta$. Thus $C'(\alpha) \geq 0$, for all $0 \leq \alpha \leq \beta$.

(ii) when $C_2\beta/2 < C_1 < C_2\mu(e^{\lambda\beta/\mu}-1)/\lambda$.

Since $A_1(0) > A_2(0)$ and $A_1(\beta) < A_2(\beta)$, there exists a unique $\alpha^*, 0 < \alpha^* < \beta$, which satisfies that $C'(\alpha) = 0$, and $C(\alpha)$ is minimized at this α^*.

(iii) when $C_1 \geq C_2\mu(e^{\lambda\beta/\mu}-1)/\lambda$.

For any $0 \leq \alpha \leq \beta$,

$$\begin{aligned} A_1(\alpha) &= (\lambda C_1 + C_2\mu)e^{-\alpha\lambda/\mu} \\ &\geq C_2\mu e^{\lambda(\beta-\alpha)/\mu} \\ &\quad \text{from the condition that } C_1 \geq C_2\mu(e^{\lambda\beta/\mu} - 1)/\lambda \\ &\geq C_2\mu(\lambda(\beta-\alpha)/\mu + 1) \\ &\quad \text{Since } e^x \geq x + 1 \text{ for } x \in R \\ &= -C_2\alpha\lambda + C_2\beta\lambda + C_2\mu \\ &\geq -C_2\alpha\lambda/2 + C_2\beta\lambda/2 + C_2\mu \\ &= A_2(\alpha). \end{aligned}$$

Thus $C'(\alpha) \leq 0$, for all $0 \leq \alpha \leq \beta$.

References

[1] Baxter, L. A. and Lee, E. Y.(1987), *An Inventory with Constant Demand and Poisson Restocking*, Probability in the Engineering and Informational Sciences, 1, 203-210.

[2] Colton, D. (1988). *Partial Differential Equations*, New York: Random House.

DEPARTMENT OF MATHEMATICS AND STATISTICS, WRIGHT STATE UNIVERSITY, DAYTON, OHIO 45435, U.S.A.