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EXISTENCE OF SOLUTIONS FOR SINGULAR NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Introduction

In this paper we study existence questions of solutions for the singular nonlinear second-order boundary value problem

$$\begin{aligned} (p(x)y'(x))' &= q(x)f(x,y(x),p(x)y'(x)) \quad \text{on} \quad (0,1) \\ y(0) = -y(1) & \quad \lim_{x\to 0^+} p(x)y'(x) = -p(1)y'(1). \end{aligned} \tag{1}$$

The problem may be singular because $p(0) = 0$ is allowed and q is not assumed to be continuous at 0. The idea of considering such problems was motivated by [2-4]. Our analysis consists in determining *a priori* bounds on all solutions to related one-parameter family of problems and applying the topological transversality theorem of Granas [4], which relies on the notion of an essential map. By a solution we shall mean a function of class $C([0,1]) \cap C^2((0,1))$ that satisfies (1). Throughout this paper we assume that $p \in C^1(0,1], q \in C(0,1]$, $p,q > 0$ on $(0,1], q, 1/p \in L^1(0,1)$, and f continuous on $[0,1] \times (-\infty,\infty) \times (-\infty,\infty)$.

A Priori Bounds on y_λ.

We consider the family of problems

$$\begin{aligned} (p(x)y_\lambda'(x))' &= \lambda q(x)f(x,y_\lambda(x),p(x)y_\lambda'(x)) \quad \text{on} \quad (0,1) \\ y_\lambda(0) = -y_\lambda(1), & \quad \lim_{x\to 0^+} p(x)y_\lambda'(x) = -p(1)y_\lambda'(1), \end{aligned} \tag{2}$$

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indexed by the parameter $\lambda \in [0,1]$.

Lemma 1. *Let pq be bounded and let there exist a constant $M > 0$ and a differentiable function $g > 0$ on $[M, \infty)$ such that $f(x,y,0) < 0$ on $(0,1) \times (-\infty, -M] \cup [M, \infty)$ and $-g(y) \le f(x,y,z)$ for $y \ge M$. Define $G(\xi) = \frac{\xi - M}{\sqrt{\int_m^\xi g(\eta)d\eta}}$ for $\xi > M$ and $G_0 = \sqrt{2\max qp}\int_0^1 \frac{dt}{p(t)}$. Then*

(a) $\lim_{\xi\to\infty} G(\xi) > G_0$, *then any solution y_λ of (2), independently of λ, satisfies $|y_\lambda(x)| \le Y$, $x \in [0,1]$, for a constant Y.*

(b) *if $\lim_{\xi\to\infty} G(\xi) = 0$ and $G(\hat{\xi}) > G_0$, then there exists an interval (ξ_1, ξ_2) such that no solution of (2) has its maximum value or absolute value of its minimum on (ξ_1,ξ_2) and $\xi_1 < \hat{\xi} < \xi_2$, where $\hat{\xi}$ is a zero greater than M of the equation*

$$2\int_M^\xi g(\eta)d\eta = (\xi - M)g(\xi). \tag{3}$$

Proof. If $\lambda = 0$, then the unique solution is $y_0 \equiv 0$. Henceforth we assume $\lambda \in (0,1]$. Let y_λ be a solution for which y_λ has an interior maximum $y_\lambda(x_0) > M$ at $x_0 \in (0,1)$. Since $f(x,y,0) < 0$ for $y > M$, y_λ has neither a local minimum greater than M nor an inflection point with a horizontal tangent and a value greater than M. From the boundary condition $y_\lambda(0) = -y_\lambda(1)$, one end point has a nonpositive value. Thus there exists an interval $(\hat{x}, x_0)$ or $(x_0, \hat{x})$ satisfying $y_\lambda(\hat{x}) = M$ and $y_\lambda{}'(x)$ a fixed sign there. On $(\hat{x}, x_0)$, we have $y_\lambda{}'(x) > 0$ and

$$-\lambda qpg(y_\lambda)y_\lambda{}' \le (py_\lambda{}')'py_\lambda{}'.$$

Integration on $(x, x_0) \subset (\hat{x}, x_0)$ and the boundedness of pq yield

$$\sqrt{2maxqp}\frac{1}{p(x)} \ge \frac{y_\lambda'(x)}{\sqrt{\int_M^{y_\lambda(x_0)} g(\eta)d\eta}}.$$

From another integration on $(\hat{x}, x_0)$, we obtain

$$G(y_\lambda(x_0)) \le G_0. \tag{4}$$

In the same manner we have (4) on $(x_0, \hat{x})$. If $\lim_{\xi\to\infty} G(\xi) > G_0$, then any interior maximum is bounded by a constant Y. Suppose $\lim_{\xi\to\infty} G(\xi) = 0$. Since $\lim_{\xi\to M} G(\xi) = 0$ and $\lim_{\xi\to M} G'(\xi) = \infty$, $G'(\xi) = 0$ has at least one zero greater than M. Let $\hat{\xi}$ satisfy (3). Then $G(\hat{\xi}) > G_0$ implies that $y_\lambda(x_0)$ does not lie between ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = G_0$. Since y_λ has no interior minimum less that $-M$, we now consider an end point extremum. Suppose that a solution y_λ has the maximum at $x = 1$ or 0. If $y'_\lambda(1) > 0$, then y_λ can not achieve its minumum at $x = 0$ and $y_\lambda(1)$ is less than the absolute value of interior minimum. Thus $p(1)y'_\lambda(1) = \lim_{x\to 0} p(x)y'_\lambda(x) = 0$ for y_λ to achieve the maximum and minimum at end points. Assume $y_\lambda(1) > M$. Then there exists a point $x_0 \in (0,1)$ such that $y_\lambda(x_0) = M$ and $y'_\lambda(x) > 0$ on $(x_0, 1)$. As in the proof of interior maximum we arrive at the inequality $G(y_\lambda(1)) \leq G_0$. The corresponding assertion holds for the case $y_\lambda(0) > M$. The lemma follows.

Lemma 2. *Suppose there exists a positive constant M satisfying $yf(x, y, 0) > 0$ on $(0,1] \times (-\infty, -M] \cup [M, \infty)$. Then for any solution y_λ of (2), $\lambda \in [0,1]$, $|y_\lambda(x)| \leq M$ for x in $[0,1]$.*

Proof. If a solution y_λ of (2) has a local maximum at $x_0 \in (0,1)$, then $y_\lambda(x_0) \leq M$, and y_λ has no local minimum less than $-M$. Suppose y_λ has the maximum and minumum at end points. As shown in the proof of Lemma 1, $\lim_{x\to 0} p(x)y'_\lambda(x) = p(1)y'_\lambda(1) = 0$. If $y_\lambda(1)$ is the maximum greater than M, then $\lim_{x\to 1} y_\lambda''(x) > 0$. This implies that y_λ is decreasing near $x = 1$. Contradiction. Similarly the minimum less than $-M$ does not occur at $x = 1$. This implies that $|y_\lambda(1)| = |y_\lambda(0)| \leq M$.

A Priori **Bounds on** py'_λ.

Lemma 3. *Let y_λ be a solution of (2) that satisfies $|y_\lambda| \leq Y$ for some constant Y and let f satisfy*

(a) $|f(x, y, z)| \leq h(|z|)$ *on* $[0,1] \times [-Y, Y] \times (-\infty, \infty)$, *where $h(z)$ is a continuous function on* $[0, \infty)$ *and*

(b) $\int_0^\infty \frac{dz}{h(z)} dz > \int_0^1 q(x)dx$ *or*

$\int_0^\infty \frac{z}{h(z)} dz > 2 \max p(x)q(x)Y$ *if pq is bounded.*

Then there exists a constant Z such that $\sup_{(0,1)} |p(x)y'_\lambda(x)| \leq Z$.

Proof. y_λ is monotone or $y'_\lambda(x_0) = 0$ for some x_0. Considering monotone case first, we have on $(0,1)$ that

$$(|py'_\lambda|)' \leq |(py'_\lambda)'| \leq q(x)h(|py'_\lambda|).$$

Multiplication by $1/h(|py'_\lambda|)$ and integration over $(0,x) \subset (0,1)$ yields

$$\int_0^{|py'_\lambda(x)|} \frac{dz}{h(z)} \leq \int_0^1 q(x)dx \tag{5}$$

since $\lim_{x\to 0} p(x)y'_\lambda(x) = 0$. Now suppose y'_λ vanishes at some point x_0. Then every $x \in [0,1]$ where $y'_\lambda \neq 0$ belongs to an interval (x,x_0) or (x_0,x) such that y'_λ has a fixed sign there. Similarly we obtain (5) again. If pq bounded, by multiplying $|py'_\lambda|/h(|py'_\lambda|)$ instead of $1/h(|py'_\lambda|)$ we have

$$\int_0^{|py'_\lambda(x)|} \frac{z}{h(z)}dz \leq 2\max pqY.$$

The result follows.

Existence of Solutions

We shall prove the existence of solutions of (1) separately for the cases (a) and (b) in Lemma 1.

Theorem 1. *Let there exist constants Y and Z such that any solution y_λ of (2) satisfies $\max_{[0,1]}|y_\lambda(x)| \leq Y$ and $\sup_{(0,1)}|p(x)y'_\lambda(x)| \leq Z$, $0 \leq \lambda \leq 1$. Then the problem (1) has a solution.*

Proof. From the differential equation itself and the continuity of f it follows that

$$\sup_{(0,1)}\left|\frac{(p(x)y'_\lambda(x))'}{q(x)}\right| \leq N \equiv \sup_{[0,1]\times[-Y,Y]\times[-Z,Z]} |f(x,y,z)|.$$

For appropriate functions v define

$$\begin{aligned}\|v\|_0 &= \max_{[0,1]}|v(x)|, \quad \|v\|_1 = \max\Big(\|v\|_0, \sup_{(0,1)}|p(x)v'(x)|\Big),\\ \|v\|_2 &= \max\Big(\|v\|_1, \sup_{(0,1)}|(p(x)v'(x))'/q(x)|\Big).\end{aligned}$$

Then we have the Banach spaces $(B, \|\cdot\|_0) = \{v \in C(0,1) : \|v\|_0 < \infty\}$, $(B_1, \|\cdot\|_1) = \{v \in C[0,1] \cap C^1(0,1) : \|v\|_1 < \infty\}$, and $(B, \|\cdot\|_2) = \{v \in C[0,1] \cap C^2(0,1) : \|v\|_2 < \infty\}$ and set a convex subset $\widehat{B_2} = \{v \in B_2 : v(0) = -v(1), \lim_{x\to 0} p(x)v'(x) = -p(1)v'(1)\}$. Define the mappings $F_\lambda : B_1 \to B$ by $(F_\lambda v)(x) = \lambda f(x, v(x), p(x)v'(x))$, $j : \widehat{B_2} \to B_1$ by $jv = v$, and $L : \widehat{B_2} \to B$ by $(Lv)(x) = (p(x)v'(x))'/q(x)$. Clearly F_λ

is continuous. Let Ω be a bounded set in $\widehat{B_2}$. Then $j\Omega$ is uniformly bounded and equicontinuous and the Arzela-Ascoli theorem implies that j is completely continuous. Now we claim that L^{-1} exists and is continuous. The solution $v \in \widehat{B_2}$ of $Lv = u$ for $u \in B$ is given uniquely by

$$\begin{aligned} v(x) &= \int_0^x \frac{1}{p(t)} \int_0^t q(s)u(s)dsdt \\ &\quad + \int_0^1 q(t)u(t)dt\Big[\frac{1}{4}\int_0^1 \frac{dt}{p(t)} - \frac{1}{2}\int_0^x \frac{dt}{p(t)}\Big] \\ &\quad -\frac{1}{2}\int_0^1 \frac{1}{p(t)} \int_0^t q(s)u(s)dsdt. \end{aligned}$$

Hence L is one to one and onto. Since $\|Lv\|_0 \le \|v\|_2$, by the Bounded Inverse Theorem L^{-1} is a continuous linear operator.

Let

$$V \equiv \{v \in \widehat{B_2} : \|v\|_2 < \max(Y, Z, N) + 1\}.$$

Then V is an open subset of the convex subset $\widehat{B_2}$ of the Banach space B_2. Now we define our compact homotopy $H_\lambda : \overline{V} \to \widehat{B_2}$ by $H_\lambda v = L^{-1}F_\lambda jv$. H_λ is fixed point free on ∂V by the construction of V. Since H_0 is a constant map and thus essential, it follows by the topological transversality theorem that H_1 is essential, i.e. (1) has a solution.

Our last theorem shows that the existence of such an interval [(b) in Lemma 1] is sufficient for us to apply the topological transversality theorem.

Theorem 2. *Let the following hypotheses hold:*

(H1) *There exists an interval (ξ_1, ξ_2) independently of $\lambda \in [0,1]$, such that no solution y_λ of (2) has the maximum value of $|y_\lambda|$ on (ξ_1, ξ_2).*

(H2) *For any solution of (2) satisfying $|y_\lambda| \le Y$, $\xi_1 < Y < \xi_2$, there exists a constant Z such that $\sup_{(0,1)}|p(x)y_\lambda'(x)| \le Z$.*

Then (1) has a solution.

Proof. The proof closely parallels that of Theorem 1 with replacement of $\|v\|_1, \|v\|_2$, and V by

$$\begin{aligned} \|v\|_1 &= \max\Big(\|v\|_0/Y, \sup_{(0,1)}|p(x)v'(x)|/Z\Big), \\ \|v\|_2 &= \max\Big(\|v\|_1/Y, \sup_{(0,1)}\Big|\frac{(p(x)v'(x))'}{q(x)}\Big|\frac{1}{N}\Big), \\ V &= \{v \in \widehat{B_2} : \|v\|_2 < 1 + \varepsilon\} \end{aligned}$$

for ϵ small enough so that $Y(1+\epsilon) < \xi_2$. Since Z and N have the property that $\sup_{(0,1)}|p(x)v'(x)| \leq Z$ and $\sup_{(0,1)}|(p(x)v'(x))'/q(x)| \leq N$, for any solution y_λ of (2) satisfying $|y_\lambda| \leq Y$, $\xi_1 < Y < \xi_2$, it follows that no solution lies on ∂V, i.e. H_λ has no fixed points on ∂V.

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