Kyungpook Mathematical Journal
Volume 31, Number 1, June, 1991

CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH THE SECOND COEFFICIENT

Sang Hun Lee and Shigeyoshi Owa

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$.
A function $f(z) \in S$ is said to be a member of the class $S(\alpha, \beta)$, which is the class of starlike functions of order $\alpha\,(0 \leq \alpha < 1)$ and type $\beta\,(0 < \beta \leq 1)$ if and only if

$$\left|\left\{\frac{zf'(z)}{f(z)} - 1\right\}\Big/\left\{\frac{zf'(z)}{f(z)} + (1-2\alpha)\right\}\right| < \beta \quad (z \in U). \tag{1.2}$$

Moreover, a function $f(z) \in S$ is in $C(\alpha, \beta)$, the class of convex functions of order $\alpha\,(0 \leq \alpha < 1)$ and type $\beta\,(0 < \beta \leq 1)$ if and only if $zf'(z) \in S(\alpha, \beta)$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.3}$$

We denote by $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ the classes obtained by taking intersections, respectively, of the classes $S(\alpha, \beta)$ and $C(\alpha, \beta)$ with T.

Received December 8, 1990.
Revised December 27, 1990.
The first author was partially supported by the Basic Science Research Institute Program Ministry of Education, 1990.

For these classes $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$, Gupta and Jain [3] showed the following lemmas.

Lemma 1. *A function $f(z)$ defined by (1.3) is in the class $S^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\alpha)\}a_n \le 2\beta(1-\alpha). \tag{1.4}$$

This result is sharp.

Lemma 2. *A function $f(z)$ defined by (1.3) is in the class $C^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(n+1-2\alpha)\}a_n \le 2\beta(1-\alpha). \tag{1.5}$$

This result is sharp.

In view of Lemma 1 and Lemma 2, we can see that $f(z)$ defined by (1.3) in $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ satisfy

$$a_2 \le \frac{2\beta(1-\alpha)}{1-2\alpha\beta+3\beta} \tag{1.6}$$

and

$$a_2 \le \frac{\beta(1-\alpha)}{1-2\alpha\beta+3\beta}, \tag{1.7}$$

respectively. We note by $S_p^*(\alpha,\beta)$ functions in the class $S^*(\alpha,\beta)$ of the form

$$f(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \ge 0) \tag{1.8}$$

and by $C_p^*(\alpha,\beta)$ functions in the class $C^*(\alpha,\beta)$ of the form

$$f(z) = z - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \ge 0), \tag{1.9}$$

where $0 \le p \le 1$.

Silverman [5], Silverman and Silvia [6], Al-Amiri [1], Finkelstein [2], Netanyahu [4], Suffridge [7] and Tepper [8] gave many intersting results for various subclasses of univalent functions with a fixed second coefficient.

In this paper, we introduce the two subclasses $S_p^*(\alpha,\beta)$ of $S^*(\alpha,\beta)$ and $C_p^*(\alpha,\beta)$ of $C^*(\alpha,\beta)$ consisting of functions with the fixed second coefficient a_2. The object of the present paper is to prove some results for convex linear combinations and some distortion theorems for functions in these classes, and to find the order of starlikeness of function in the class $C_p^*(\alpha,\beta)$.

2. Convex Linear Combinations

In this section, we show that the classes $S_p^*(\alpha,\beta)$ and $C_p^*(\alpha,\beta)$ are closed under convex linear combinations.

Theorem 1. *Let*

$$f_2(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 \tag{2.1}$$

and

$$f_n(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \frac{2(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}z^n \tag{2.2}$$

for $n = 3, 4, \cdots$. *Then* $f(z)$ *is in the class* $S_p^*(\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{2.3}$$

where $\lambda_n \geq 0$ *and* $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. We assume that the function $f(z)$ can be expressed in the from (2.3). Then we have

$$\begin{aligned} f_n(z) &= z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \sum_{n=3}^{\infty} \frac{2\lambda_n(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}z^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n, \end{aligned} \tag{2.4}$$

where

$$A_2 = \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} \tag{2.5}$$

and

$$A_n = \frac{2\lambda_n(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \quad (n = 3, 4, \cdots) \tag{2.6}$$

Hence we can see that

$$\sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\} A_n \tag{2.7}$$

coefficient a_2. The object of the present paper is to prove some results for convex linear combinations and some distortion theorems for functions in these classes, and to find the order of starlikeness of function in the class $C_p^*(\alpha, \beta)$.

2. Convex Linear Combinations

In this section, we show that the classes $S_p^*(\alpha, \beta)$ and $C_p^*(\alpha, \beta)$ are closed under convex linear combinations.

Theorem 1. *Let*

$$f_2(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} z^2 \tag{2.1}$$

and

$$f_n(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} z^2 - \frac{2(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} z^n \tag{2.2}$$

for $n = 3, 4, \cdots$. Then $f(z)$ is in the class $S_p^*(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{2.3}$$

where $\lambda_n \geq 0$ and $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. We assume that the function $f(z)$ can be expressed in the from (2.3). Then we have

$$\begin{aligned} f_n(z) &= z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} z^2 - \sum_{n=3}^{\infty} \frac{2\lambda_n(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} z^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n, \end{aligned} \tag{2.4}$$

where

$$A_2 = \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} \tag{2.5}$$

and

$$A_n = \frac{2\lambda_n(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \quad (n = 3, 4, \cdots) \tag{2.6}$$

Hence we can see that

$$\sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\} A_n \tag{2.7}$$

$$
\begin{aligned}
&= 2\beta(1-\alpha)[p+(1-p)\sum_{n=3}^{\infty}\lambda_n\} \\
&= 2\beta(1-\alpha)\{1-\lambda_2(1-p)\} \\
&\leq 2\beta(1-\alpha).
\end{aligned}
$$

because $0 \leq p \leq 1$ and $0 \leq \lambda_2 \leq 1$. It follows from (1.4) that $f(z)$ is in the class $S_p^*(\alpha, \beta)$.

Conversely, we suppose that $f(z)$ defined by (1.8) is in the class $S_p^*(\alpha, \beta)$. With the aid of (1.4), we obtain

$$
(2.8) \quad 2p\beta(1-\alpha) + \sum_{n=3}^{\infty}\{(n-1)+\beta(n+1-2\alpha)\}a_n \leq 2\beta(1-\alpha),
$$

further

$$
(2.9) \qquad a_n \leq \frac{2(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \quad (n = 3, 4, \cdots)
$$

Putting

$$
(2.10) \qquad \lambda_n = \frac{(n-1)+\beta(n+1-2\alpha)}{2(1-p)\beta(1-\alpha)} \quad (n = 3, 4, \cdots)
$$

and

$$
(2.11) \qquad \lambda_2 = 1 - \sum_{n=3}^{\infty}\lambda_n,
$$

we have (2.3). This completes the proof of the theorem.

Theorem 2. *Let*

$$
(2.12) \qquad f_2(z) = z - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2
$$

and

$$
(2.13) \quad f_n(z) = z - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \frac{2(1-p)\beta(1-\alpha)}{n\{(n-1)+\beta(n+1-2\alpha)\}}z^n
$$

for $n = 3, 4, \cdots$. Then $f(z)$ is in the class $C_p^*(\alpha, \beta)$ if and only if it can be expressed in the form

$$
(2.14) \qquad f(z) = \sum_{n=2}^{\infty}\lambda_n f_n(z),
$$

where $\lambda_n \geq 0$ and $\sum_{n=2}^{\infty}\lambda_n = 1$.

The proof of Theorem 2 from the bounds in (1.5) just as Theorem 1 followed from the bounds in (1.4).

3. Distortion Theorems

In order to show the distortion for $f(z)$ in $S_p^*(\alpha, \beta)$, we need the following lemmas.

Lemma 3. *Let the function $f_3(z)$ be defined by*

$$f_3(z) = z - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}z^3. \tag{3.1}$$

Then, for $0 \le r < 1$ and $0 \le p \le 1$,

$$|f_3(re^{i\theta})| \ge r - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^3. \tag{3.2}$$

with equality for $\theta = 0$. For either $0 \le p < p_0$ and $0 \le r \le r_0$ or $p_0 \le p \le 1$.

$$|f_3(re^{i\theta})| \le r + \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^3. \tag{3.3}$$

with equality for $\theta = \pi$. Further, for $0 \le p < p_0$ and $r_0 \le r < 1$,

$$\begin{aligned}|f_3(re^{i\theta})| \le r\Big\{&\Big(1 + \frac{p^2\beta(1-\alpha)(1-\alpha\beta+2\beta)}{(1-p)(1-2\alpha\beta+3\beta)^2}\Big)\\ &+2\beta(1-\alpha)\Big(\frac{1-p}{1-\alpha\beta+2\beta} + \frac{p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^2\\ &+\frac{(1-p)\beta^2(1-\alpha)^2}{1-\alpha\beta+2\beta}\Big(\frac{1-p}{1-\alpha\beta+2\beta} + \frac{p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^4\Big\}^{\frac{1}{2}}\end{aligned} \tag{3.4}$$

with equality for

$$\theta = \cos^{-1}\Big(\frac{p(1-p)\beta(1-\alpha)r^2 - p(1-\alpha\beta+2\beta)}{2(1-p)(1-2\alpha\beta+3\beta)r}\Big) \tag{3.5}$$

where

$$\begin{aligned}p_0 &= \frac{1}{2\beta(1-\alpha)}\{-(3-4\alpha\beta+7\beta)^2\\ &\quad+\sqrt{(3-4\alpha\beta+7\beta)^2 + 8\beta(1-\alpha)(1-2\alpha\beta+3\beta)}\}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}r_0 = &\frac{1}{p(1-p)\beta(1-\alpha)}\{-(1-p)(1-2\alpha\beta+3\beta)\\ &+\sqrt{(1-p)^2(1-2\alpha\beta+3\beta)^2 + p^2(1-p)\beta(1-\alpha)(1-2\alpha\beta+2\beta)}\}\end{aligned} \tag{3.7}$$

Proof. We employ the same technique as used by Silverman and Silvia [6]. By the simple computation, we have

$$(3.8)\qquad \frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = 4\beta(1-\alpha)^3 r^3 \sin\theta\Big\{\frac{p}{1-2\alpha\beta+3\beta} - \frac{p(1-p)\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)(1-\alpha\beta+2\beta)}r^2 + \frac{2(1-p)}{1-\alpha\beta+2\beta}r\cos\theta\Big\}$$

further $\partial|f_3(re^{i\theta})|^2/\partial\theta = 0$ for $\theta_1 = 0$, $\theta_2 = \pi$ and

$$(3.9)\qquad \theta_3 = \cos^{-1}\Big(\frac{p(1-p)\beta(1-\alpha)r^2 - p(1.-\alpha\beta+2\beta)}{2(1-p)(1-2\alpha\beta+3\beta)r}\Big)$$

Since θ_3 is a valid root only when

$$(3.10)\qquad \Big|\frac{p(1-p)\beta(1-\alpha)r^2 - p(1-\alpha\beta+2\beta)}{2(1-p)(1-2\alpha\beta+3\beta)r}\Big| \le 1$$

we have a third root if and only if $r_0 \le r < 1$ and $0 \le c \le c_0$. Thus we have the lemma by comparing the extremal values $|f_3(re^{i\theta}k)|$ $(k = 1, 2, 3)$ on the appropriate intervals.

Lemma 4. *Suppose $n \ge 4$. Let the function $f_n(z)$ be defined by (2.2). Then*

$$(3.11)\qquad |f_n(re^{i\theta})| \le |f_4(-r)|.$$

Proof. Since $2p(1-p)\beta(1-\alpha)r^n/\{(n-1)+\beta(n+1-2\alpha)\}$ is a decreasing function of n, we obtain

$$(3.12)\qquad \begin{aligned} |f_n(re^{i\theta})| &\le r + \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{2(1-p)\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)}r^n \\ &\le r + \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{2(1-p)\beta(1-\alpha)}{3-2\alpha\beta+5\beta}r^4 \\ &= -f_4(-r) \end{aligned}$$

which shows (3.11).

Theorem 3. *Let the function $f(z)$ defined by (1.8) be in the class $S_p^*(\alpha,\beta)$. Then, for $0 \le r < 1$,*

$$(3.13)\qquad |f(re^{i\theta})| \ge r - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^3$$

with equality for the function $f_3(z)$ *at* $z=r$, *and*

$$(3.14) \qquad |f(re^{i\theta})| \leq Max\{Max_\theta|f_3(re^{i\theta})|, -f_4(-r)\},$$

where $Max_\theta|f_3(re^{i\theta})|$ *is given by Lemma 3.*

The proof of Theorem 3 follows from Lemma 3 and Lemma 4.

Lemma 5. *Let the function* $f_3(z)$ *be defined by (3.1). Then, for* $0 \leq r < 1$ *and* $0 \leq p \leq 1$,

$$(3.15) \qquad |f_3'(re^{i\theta})| \geq 1 - \frac{4p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{3(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^2$$

with equality for $\theta = 0$. *For eithere* $0 \leq p < p_1$ *and* $0 \leq r \leq r_1$ *or* $p_0 \leq p \leq 1$,

$$(3.16) \qquad |f_3'(re^{i\theta})| \leq 1 + \frac{4p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{3(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^2$$

with equality for $\theta = \pi$. *Further, for* $0 \leq p < p_1$ *and* $r_1 \leq r < 1$,

$$(3.17) \qquad |f_3'(re^{i\theta})| \leq \Big\{\Big(1 + \frac{4p^2\beta(1-\alpha)(1-\alpha\beta+2\beta)}{3(1-p)(1-2\alpha\beta+3\beta)^2}\Big)$$
$$+2\beta(1-\alpha)\Big(\frac{3(1-p)}{1-\alpha\beta+2\beta} + \frac{4p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^2$$
$$+\frac{3(1-p)\beta^2(1-\alpha)^2}{1-\alpha\beta+2\beta}\Big(\frac{3(1-p)}{1-\alpha\beta+2\beta} + \frac{4p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^4\Big\}^{\frac{1}{2}}$$

with equality for

$$(3.18) \qquad \theta = \cos^{-1}\Big(\frac{3p(1-p)\beta(1-\alpha)r^2 - p(1-\alpha\beta+2\beta)}{3(1-p)(1-2\alpha\beta+3\beta)r}\Big)$$

where

$$(3.19) \quad p_1 = \frac{1}{3\beta(1-\alpha)}\{-2(1-\alpha\beta+2\beta)$$
$$+\sqrt{(4(1-\alpha\beta+2\beta)^2 + 9\beta(1-\alpha)(1-2\alpha\beta+3\beta)}\}$$

and

$$(3.20) \quad r_1 = \frac{1}{6p(1-p)\beta(1-\alpha)}\{-3(1-p)(1-2\alpha\beta+3\beta)$$
$$+\sqrt{9(1-p)^2(1-2\alpha\beta+3\beta)^2 + 12p^2(1-p)\beta(1-\alpha)(1-\alpha\beta+2\beta)}\}$$

The proof of Lemma 5 is given in much the same way as Lemma 3.

Theorem 4. *Let the function $f(z)$ defined by (1.8) be in the class $S_p^*(\alpha,\beta)$. Then, for $0 \le r < 1$,*

$$(3.21) \qquad |f'(re^{i\theta})| \ge 1 - \frac{4p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{3(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^2$$

with equality for $f_3'(z)$ at $z = r$, and

$$(3.22) \qquad |f'(re^{i\theta})| \le Max\{Max_\theta|f_3'(re^{i\theta})|, f_4'(-r)\},$$

where $Max_\theta|f_3'(re^{i\theta)})|$ is given by Lemma 5.

In the same way, we can show the following results for functions $f(z)$ in $C_p^*(\alpha,\beta)$.

Lemma 6. *Let the function $f_3(z)$ be defined by*

$$(3.23) \qquad f_3(z) = z - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}z^2 - \frac{(1-p)\beta(1-\alpha)}{3(1-\alpha\beta+2\beta)}z^3.$$

Then, for $0 \le r < 1$ and $0 \le p \le 1$,

$$(3.24) \qquad |f_3(re^{i\theta})| \ge r - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{3(1-\alpha\beta+2\beta)}r^3.$$

with equality for $\theta = 0$. For either $0 \le p < p_0'$ and $0 \le r \le r_0'$ or $p_0' \le p \le 1$.

$$(3.25) \qquad |f_3(re^{i\theta})| \le r + \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{3(1-\alpha\beta+2\beta)}r^3.$$

with equality for $\theta = \pi$. Further, for $0 \le p < p_0'$ and $r_0' \le r < 1$,

$$(3.26) \qquad |f_3(re^{i\theta})| \le r\Big\{\Big(1 + \frac{9p^2\beta(1-\alpha)(1-\alpha\beta+2\beta)}{8(1-p)(1-2\alpha\beta+3\beta)^2}\Big) + \beta(1-\alpha)\Big(\frac{2(1-p)}{3(1-\alpha\beta+2\beta)} + \frac{p^2\beta(1-\alpha)}{2(1-2\alpha\beta+3\beta)^2}\Big)r^2 + \frac{(1-p)\beta^2(1-\alpha)^2}{3(1-\alpha\beta+2\beta)}\Big(\frac{(1-p)}{3(1-\alpha\beta+2\beta)} + \frac{p^2\beta(1-\alpha)}{4(1-2\alpha\beta+3\beta)^2}\Big)r^4\Big\}^{\frac{1}{2}}$$

with equality for

$$(3.27)\qquad \theta = \cos^{-1}\left(\frac{p(1-p)\beta(1-\alpha)r^2 - 3p(1-\alpha\beta+2\beta)}{4(1-p)(1-2\alpha\beta+3\beta)r}\right),$$

where

$$(3.28)\ p_0' = \frac{1}{2\beta(1-\alpha)}\{-(7-10\alpha\beta+17\beta) + \sqrt{(7-10\alpha\beta+17\beta)^2+16\beta(1-\alpha)(1-2\alpha\beta+3\beta)}\}$$

and

$$(3.29)\quad r_0' = \frac{1}{p(1-p)\beta(1-\alpha)}\{-2(1-p)(1-2\alpha\beta+3\beta) + \sqrt{4(1-p)^2(1-2\alpha\beta+3\beta)^2+3p^2(1-p)\beta(1-\alpha)(1-\alpha\beta+2\beta)}\}$$

Theorem 5. *Let the function $f(z)$ defined by (1.9) be in the class $C_p^*(\alpha,\beta)$. Then, for $0 \le r < 1$,*

$$(3.30)\qquad |f(re^{i\theta})| \ge r - \frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r^2 - \frac{(1-p)\beta(1-\alpha)}{3(1-\alpha\beta+2\beta)}r^3$$

with equality for the function $f_3(z)$ at $z = r$, and

$$(3.31)\qquad |f(re^{i\theta})| \le Max\{Max_\theta|f_3(re^{i\theta})|, -f_4(-r)\},$$

where $Max_\theta|f_3(re^{i\theta})|$ is given by Lemma 6.

Lemma 7. *Let the function $f_3(z)$ be defined by (3.23). Then, for $0 \le r < 1$, and $0 \le p < 1$,*

$$(3.32)\qquad |f'(re^{i\theta})| \ge 1 - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^2$$

with equality for $\theta = 0$. For either $0 \le p < p_1'$ and $0 \le r \le r_1'$ or $p_1' \le p \le 1$,

$$(3.33)\qquad |f_3'(re^{i\theta})| \le 1 + \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{(1-p)\beta(1-\alpha)}{1-\alpha\beta+2\beta}r^2$$

with equality for $\theta = \pi$. *Further, for* $0 \leq p < p_1'$ *and* $r_1' \leq r < 1$,

$$(3.34)\qquad |f_3'(re^{i\theta})| \leq \Big\{\Big(1 + \frac{p^2\beta(1-\alpha)(1-\alpha\beta+2\beta)}{(1-p)(1-2\alpha\beta+3\beta)^2}\Big) + 2\beta(1-\alpha)\Big(\frac{(1-p)}{1-\alpha\beta+2\beta} + \frac{p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^2 + \frac{(1-p)\beta^2(1-\alpha)^2}{1-\alpha\beta+2\beta}\Big(\frac{(1-p)}{1-\alpha\beta+2\beta} + \frac{p^2\beta(1-\alpha)}{(1-2\alpha\beta+3\beta)^2}\Big)r^4\Big\}^{\frac{1}{2}}$$

with equality for

$$(3.35)\qquad \theta = \cos^{-1}\Big(\frac{p(1-p)\beta(1-\alpha)r^2 - p(1-\alpha\beta+2\beta)}{2(1-p)(1-2\alpha\beta+3\beta)r}\Big),$$

where

$$(3.36)\quad p_1' = \frac{1}{2\beta(1-\alpha)}\{-(3-4\alpha\beta+7\beta) + \sqrt{((3-4\alpha\beta+7\beta)^2 + 8\beta(1-\alpha)(1-2\alpha\beta+3\beta)}\}$$

and

$$(3.37)\quad r_1' = \frac{1}{p(1-p)\beta(1-\alpha)}\{-(1-p)(1-2\alpha\beta+3\beta) + \sqrt{(1-p)^2(1-2\alpha\beta+3\beta)^2 + p^2(1-p)\beta(1-\alpha)(1-\alpha\beta+2\beta)}\}$$

Theorem 6. *Let the function* $f(z)$ *defined by (1.9) be in the class* $C_p^*(\alpha,\beta)$. *Then, for* $0 \leq r < 1$,

$$(3.38)\qquad |f'(re^{i\theta})| \geq 1 - \frac{2p\beta(1-\alpha)}{1-2\alpha\beta+3\beta}r - \frac{(1-p)\beta(1-\alpha)}{(1-\alpha\beta+2\beta)}r^2$$

with equality for the function $f_3'(z)$ *at* $z = r$, *and*

$$(3.39)\qquad |f'(re^{i\theta})| \leq Max\{Max_\theta|f_3'(re^{i\theta})|, f_4'(-r)\},$$

where $Max_\theta|f_3'(re^{i\theta})|$ *is given by Lemma 7.*

4. Starlikeness of $C_p^*(\alpha,\beta)$

Finally, we consider about the starlikeness for functions in the class $C_p^*(\alpha,\beta)$.

Theorem 7. *Let the function* $f(z)$ *defined by (1.9) be in the class* $C_p^*(\alpha,\beta)$. *Then* $f(z)$ *is in the class* $S_p^*(\alpha_1,\beta_1)$ *for* α_1 $(0 \le \alpha_1 < 1)$ *and* β_1 $(0 < \beta_1 \le 1)$ *which satisfy*

$$\begin{aligned}(4.1)\qquad & 2\{3A+(1-p)B\}\alpha_1\beta_1 - \{9A+4(1-p)B\}\beta_1 \\ & -\{3A+2(1-p)B-6AB\}=0,\end{aligned}$$

where $A = 1-\alpha\beta+2\beta$ *and* $B = 1-2\alpha\beta+3\beta$. *This result is sharp for the function* $f_3(z)$ *defined by (3.23).*

Proof. *With the condition (4.1), we can see that*

$$\begin{aligned}(4.2)\qquad & (1-2\alpha_1\beta_1+3\beta_1)\frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} \\ & +2(1-\alpha_1\beta_1+2\beta_1)\frac{(1-p)\beta(1-\alpha)}{3(1-\alpha\beta+2\beta)} = 2\beta_1(1-\alpha_1)\end{aligned}$$

so that $f_3(z) \in S_p^*(\alpha_1,\beta_1)$. *Next, let* $f(z) \in C_p^*(\alpha,\beta)$. *In view of Theorem 2, we may set*

$$(4.3)\qquad a_n = \frac{2\lambda_n(1-p)\beta(1-\alpha)}{n\{(n-1)+\beta(n+1-2\alpha)\}} \quad (n=3,4,\cdots)$$

and

$$(4.4)\qquad \sum_{n=3}^{\infty}\lambda_n \le 1.$$

Since $\{(n-1)+\beta_1(n+1-2\alpha_1)\}/n\{(n-1)+\beta(n+1-2\alpha)\}$ *is a decreasing function of* n, *the expression*

$$2\beta(1-\alpha)\sum_{n=3}^{\infty}\frac{\lambda_n\{(n-1)+\beta_1(n+1-2\alpha_1)\}}{n\{(n-1)+\beta(n+1-2\alpha)\}}$$

is maximized when $\lambda_3 = 1$, *further* $f(z) = f_3(z)$. *Consequently we obtain*

$$\begin{aligned}(4.5)\qquad & (1-2\alpha_1\beta_1+3\beta_1)\frac{p\beta(1-\alpha)}{1-2\alpha\beta+3\beta} \\ & +2\beta(1-\alpha)\sum_{n=3}^{\infty}\frac{\lambda_n\{(n-1)+\beta_1(n+1-2\alpha_1)\}}{n\{(n-1)+\beta(n+1-2\alpha)\}} \\ & \le 2\beta_1(1-\alpha_1)\end{aligned}$$

with the aid of (4.2). This completes the proof of the theorem.

References

[1] H.S. Al-Amiri, *On p-close-to-star functions of order* α, Proc. Amer. Math. Soc., 29(1971), 103-108.

[2] M. Finkelstein, *Growth estimates of convex functions*, Proc. Amer. Math. Soc., 18(1967), 412-418.

[3] V.P. Gupta and P.K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., 14(1976), 409-416.

[4] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Rational Mech. Anal., 32(1969), 100-112.

[5] H. Silverman, *On a class of close-to-convex functions*, Proc. Amer. Math. Soc., 36(1972), 477-484.

[6] H. Silverman and E.M. Silvia, *Fixed coefficients for subclasses of starlike functions*, Houston J. Math., 7(1981), 129-136.

[7] T.J. Suffridge, *A new criterion for starlike functions*, Indiana Univ. Math. J., 28(1979), 429-443.

[8] D. Tepper, *On the radius of convexity and boundary distortion of schlicht functions*, Trans. Amer. Math. Soc., 150(1970), 519-528.

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.

DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY HIGASHI-OSAKA, OSAKA 577, JAPAN.