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A NOTE ON INFLUENCE OF SUBGROUP RESTRICTIONS IN FINITE GROUP STRUCTURE

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We consider in this note group theoretic restrictions on specific subgroups of a finite group G. These restrictions yield different characterizations of G. All the groups considered in this note are finite. We shall use a result due to R. Baer [1, lemma 3, p.12] in proving theorem 1 and we state it below for the sake of completeness.

Lemma 1. *If the group G possesses a maximal subgroup with core 1 then the following properties of G are equivalent.*

(1) *The indices in G of all the maximal subgroups with core 1 are powers of one and the same prime p.*

(2) *There exists one and only one minimal normal subgroup of G, and there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1.*

(3) *There exists a soluble normal subgroup, not 1, in G.*

Theorem 1. *If the indices of all non normal maximal subgroups of a group G are equal then G is solvable.*

Proof. If for some maximal subgroup M of G, $[G : M]_p \neq 1$ for some prime p then there exists however a maximal subgroup M^* such that $[G : M^*]_p = 1$. Therefore G is not simple and by induction G/N is solvable and N is unique. If $N \not\subseteq \phi(G)$ then $G = XN$ for some maximal subgroup X of G and X is core free. Since the indices of all core free maximal subgroups are same it now follows by lemma 1 that N is solvable which implies G is solvable.

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Proposition 1. *If a nonabelian group G has a maximal subgroup M whose intersection with any other maximal subgroup is trivial then G must be elementary abelian by cyclic.*

Proof. If M is the only maximal subgroup of G then G is cyclic and also note that G cannot be a p-group either. For if M_1 and M are two maximal subgroups of G then each one of them is normal in G and $|M_1| = |M| = p^{n-1}$ if $|G| = p^n$. This implies $|G| = p^n = |M_1| \cdot |M| = p^{2n-2}$ i.e. $n = 2$ and G is abelian. Now $M \cap M^x = < e > \forall x$ in G implies $G = FM$, $F \cap M = < e >$ where F is the Frobenius kernel. Consequently F is elementary abelian for some prime p. If M_0 is a maximal subgroup of M then FM_0 is also a maximal subgroup of G and hence $G = F < x >$, $| < x > | = q =$ a prime.

The location of the prime ordered elements and elements of order 4 in the center of a group G imply that G is nilpotent. This theorem is due to N. Ito (Thm. 5.5, p.435 [2]). The following theorem is a dual of Ito's result.

Theorem 2. *A non abelian group G in which every minimal subgroup is self centralizing is a group of order pq, p, q are different primes.*

Proof. Let x be an element of G such that $|x| = p_n$, the smallest prime divisor of $|G|$. If an element y normalizes $< x >$ then each y_i also normalizes $< x >$ where $< y > = < y_1 > \times < y_2 > \times \cdots \times < y_n >$, $|y_i| = p_i^{\alpha_i}$, $p_i =$ a prime. Therefore $< y_i >< x > = < x >< y_i >$ is supersolvable and so each y_i centralizes $x, 1 \leq i \leq n-1$. In $< y_n >< x > = < x >< y_n > = H$, $< y_n >$ is a maximal subgroup and consequently is normal in H. (If $< x > \subsetneq < y_n >$ then of course y_n trivially centralizes x). It follows that y_n and x centralize each other and we therefore conclude $< x > \cap < x >^g = < e > \forall g$ in $G \setminus < x >$. Thus $G = F < x >$, $F \cap < x > = < e >$, F is the Frobenius kernel. Evidently F is divisible by one prime and $F = < y >, | < y > | =$ a prime.

Remark. It suffices in the proof however to use self centralizing property of elements corresponding to the smallest prime divisor of $|G|$.

Proposition 2. *If the order of a group G is divisible by at least two primes and every proper subgroup is of prime power order then G is elementary abelian by cyclic.*

Proof. Evidently every Sylow p–subgroup of G is a maximal subgroup and therefore G is solvable [6]. If N is a minimal normal subgroup of G then

$G = NQ$, $(|N|, |Q|) = 1$. This however implies $G = N < x >, |x| = q$ for some prime q.

The motivation for the next theorem is derived from the fact that Sylow subgroups corresponding to same prime in a subgroup H of a group G are conjugate in H itself. It characterizes groups in which not only Sylow subgroups but prime power subgroups of same order are conjugate in each subgroup.

Theorem 3. *If subgroups of same prime power order are conjugate in any subgroup of G then G is supersolvable, the Sylow p-subgroups for $p > 2$ are cyclic and the Sylow 2-subgroup has a cyclic normal subgroup of index 2.*

Proof. By induction every maximal subgroup of G is supersolvable and so G is solvable. Let N be a minimal normal subgroup of G. G/N is supersolvable by induction and since N is elementary abelian it follows that $|N| = p$, a prime. Therefore G is supersolvable.

Now suppose X is a minimal normal subgroup of G and let $|X| = p$. Note that G has exactly one subgroup of order p. If P is a Sylow p–subgroup of G then every maximal subgroup of P has exactly one subgroup of order p and by induction is cyclic. Therefore all abelian normal subgroups of P are cyclic. By Thm. 7.5 [2, p.304] P is cyclic if $p > 2$. If $p = 2$ then P has a cyclic normal subgroup of index 2.

Consider G/X and by induction all its Sylow p–subgroups for $p > 2$ are cyclic and a Sylow 2–subgroup has a cyclic maximal subgroup of index 2. This however implies that the Sylow subgroups of G have the disired property and the theorem is proved completely.

Theorem 4. *If every minimal subgroup of a group G is complemented in G then G is supersolvable.*

Proof. Let H be any subgroup of G and $< a >$ be a minimal subgroup in H. Then $G =< a > T, < a > \cap T =< e >$. Consequently, $H =< a > (H \cap T), < a > \cap (H \cap T) =< e >$. By induction it now follows that every maximal subgroup of G is supersolvable and therefore G is solvable [5, Thm. 2.3, p.10]. Let N be a minimal normal subgroup of G. If $b \in N$ then $G =< b > K, < b > \cap K =< e >$ and K is a maximal subgroup of G. This implies $N =< b > (N \cap K)$ by Dedekind's modular law and N being minimal normal in G it follows that $N \cap K =< e >$. Hence $N =< b >$ and G/N are supersolvable which however implies that G is supersolvable.

Remark. The Sylow subgroups of such a group G as stated in the theorem are not necessarily cyclic as the instance of S_3 might suggest.

Let $G =< a, b, x | a^3 = b^3 = 1, ab = ba, a^x = a^2, b^x = b^2, x^2 = e >$. G is a group of order 18 which is supersolvable and every minimal subgroup is complemented in G. However, the Sylow 3–subgroup of G is not cyclic.

References

[1] R. Baer, *Classes of finite groups and their properties*, Illinois J. Math., Vol I(1957), 115-187.

[2] B. Huppert, *Endlichen Gruppen I*, Springer, N.Y.1967.

[3] N.P. Mukherjee, *The hyperquasicenter of a finite group II*, Proc. Amer. Soc. 32(1972), 24-28.

[4] ________, *A note on normal index and maximal subgroups in finite groups*, Illinois J. Math., 2(1975), 173-178.

[5] M. Weinstein, *Between nilpotent and solvable*, Polygonal Publishing House, NJ, 1982.

[6] J. Thompson, *Proof of a conjecture of Frobenius*, Proc. Symp. Pure Math. Amer. Math. Soc., I(1959).

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