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FIXED POINT THEOREMS IN NONARCHIMEDEAN MENGER SPACES

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Nonarchimedean probabilistic metric spaces and some topological preliminaries on them were first studied by Istrătescu and Crivăt [7] (see, also [6]). Some fixed point theorems for mappings on nonarchimedean Menger spaces have been proved by Istrătescu [4,5] as a result of the generalizations of some of the results of Sehgal and Bharucha-Reid [11] and Sherwood [12]. Achari [1] studied the fixed points of quasi-contraction type mappings in nonarchimedean probabilistic metric spaces and generalized the results of Istrătescu [5]. Recently, Singh and Pant [15] have established common fixed point theorems for weakly commuting quasi-contraction pair of mappings on nonarchimedean Menger spaces.

In the present paper we replace the condition of commutativity by that of preorbital commutativity, a condition weaker than commutativity and prove some common fixed point theorems for S-type and F-type quasi-contraction triplets (see definitions 1,4) of self-mappings on nonarchimedean Menger spaces. Extension to uniform spaces and application to product spaces of one of the results are also given.

The following definition is due to Istrătescu [5].

Definition 1. Let $F_{u,v}$ denote the value at $u, v \in X \times X$ of the function $\mathcal{F}$: $X \times X \to \mathcal{L}$, the collection of all distribution functions. A nonarchimedean Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a nonarchimedean probabilistic metric space and t is a t-norm such that the nonarchimedean triangle inequality

$$(1) \qquad F_{u,w}(\max\{x, y\}) \geq t\{F_{u,v}(x), F_{v,w}(y)\}$$

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holds for all $u, v, w \in X$ and $x, y \geq 0$. Hereafter X stands for a non archimedean Menger space.

We introduce the following :

Definition 2. Three mappings $f, g, h : X \to X$ are called an S–type quasi–contraction triplet $(f, g; h)$ (S after Singh [13]) iff there exists a constant $k \in (0,1)$ such that for every u, v in X,

(2) $F_{fu,fv}(kx) \geq \max\{F_{hu,hv}(x), F_{fu,hu}(x), F_{gv,hv}(x), F_{fu,hv}(x), F_{gv,hv}(x)\}$

holds for all $x > 0$.

On the lines of Tiwari and Singh [17] we have the following :

Definition 3. Assume that a sequence of $O(f, g; hu_0)$ converges to a point u in $h(X)$ and $B_u = \{z : hz = u\}$ which is nonempty. For a positive integer N, define $A_N = \{u_n \in A : n \geq N\}$.

Then the mapping f and h will be called $(f, g; hu_0)$ preorbitally commuting if the restrictions of f and h on $A_N \cup B_N$ are commuting for some positive integer N.

Now we introduce the following :

Definition 4. Three mappings f, g, h on a nonarchimedean Menger space $(X, \mathcal{F}, t)$ are called an F–type quasi–contraction triplet $(f; g, h)$(F after Fisher [3]) iff there exists a constant $k \in (0,1)$ such that for every u, v in X,

(3) $F_{fu,fv}(kx) \geq \max\{F_{gu,hv}(x), F_{fu,gu}(x), F_{fv,hv}(x), F_{fu,hv}(x), F_{fv,gu}(x)\}$

holds for all $x > 0$.

The following definition is also on the lines of Tiwari and Singh [17].

Definition 5. Assume that a subsequence $\{fu_n\}$ of $O(fu_0; g, h)$ converges to a point u in $g(X) \cap h(X)$ and $B_u = \{z : gz = u\} \cup \{w; hw = u\}$. Let for a positive integer N, $A_N = \{u_n \in A : n \geq N\}$.

The mappings f and g will be called $(fu_0; g, h)$ preorbitally commuting if the restrictions of f and g on $A_N \cup B_u$ are commuting for some positive integer N.

Theorem 1. *Let $(X, \mathcal{F}, t)$ be a nonarchimedean Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0,1]$ and $(f, g; h)$ an*

S-type quasi-contraction triplet of self mappings on X. If there is a point u_0 and a sequence $\{u_n\}$ in X such that

(i) $hu_{2n+1} = fu_{2n}, hu_{2n+2} = gu_{2n+1}, \quad n = 0, 1, 2, \cdots;$

(ii) *$h(X)$ is $(f, g; hu_0)$-orbitally complete;*

(iii) *h is $(f, g; hu_0)$-preorbitally commuting with f and g, then f, g and h have a unique common fixed point and $\{hu_n\}$ converges to the fixed point.*

Proof. By (1), (2) and (i)

$$\begin{aligned} F_{hu_{2n+1},hu_{2n+2}}(kx) &= F_{fu_{2n},gu_{2n+1}}(kx) \\ &\geq \max\{F_{hu_{2n},hu_{2n+1}}(x), F_{fu_{2n},hu_{2n}}(x), \\ &\quad F_{gu_{2n+1},hu_{2n+1}}(x), F_{fu_{2n},hu_{2n+1}}(x), \\ &\quad F_{gu_{2n+1},hu_{2n}}(x)\} \\ &\geq \max\{F_{hu_{2n},hu_{2n+1}}(x), F_{hu_{2n+2},hu_{2n+1}}(x), \\ &\quad F_{hu_{2n+2},hu_{2n}}(x)\} \\ &= \max\{F_{hu_{2n},hu_{2n+1}}(x), F_{hu_{2n+2},hu_{2n+1}}(x), \\ &\quad F_{hu_{2n+2},hu_{2n}}(\max\{x, kx\})\} \\ &\geq \max\{F_{hu_{2n},hu_{2n+1}}(x), F_{hu_{2n+2},hu_{2n+1}}(x), \\ &\quad F_{hu_{2n+2},hu_{2n+1}}(x), F_{hu_{2n+1},hu_{2n}}(kx)\} \\ &= F_{hu_{2n},hu_{2n+1}}. \end{aligned}$$

Therefore, by the lemma (Singh and Pant, [14]), $\{hu_n\}$ is a Cauchy sequence and by virtue of (ii) converges to a point p (say) in $h(X)$. This implies the existence of a point z in X such that $hz = p$. Now, let $U_{hz}(\varepsilon, \lambda)$ be a neighbourhood of hz. Then for $\varepsilon, \lambda > 0$, there exists an integer N such that

$$(4) \qquad F_{hu_{2n},hz}(\varepsilon/k) > 1 - \lambda \quad \text{and} \quad F_{hu_{2n},hu_{2n+1}}(\varepsilon/k) > 1 - \lambda$$

for all $n \geq N$.

Again by (1), (2) and (i),

$$\begin{aligned} F_{hu_{2n+1},gz}(\varepsilon) &= F_{fu_{2n},gz}(\varepsilon) \\ &\geq \max\{F_{hu_{2n},hz}(\varepsilon/k), F_{hu_{2n+1},hu_{2n}}(\varepsilon/k), F_{gz,hz}(\varepsilon/k), \\ &\quad F_{hu_{2n+1},hz}(\varepsilon/k), F_{gz,hu_{2n}}(\varepsilon/k)\} \\ &\geq \max\{F_{hu_{2n},hz}(\varepsilon/k), F_{hu_{2n},hu_{2n+1}}(\varepsilon/k)\} \\ &> 1 - \lambda, \text{ by } (4). \end{aligned}$$

So, $gz = hz$. Similarly $fz = hz$. Thus $fz = gz = hz = p$. Now by (iii),

$$fhz = hfz = ffz = hhz \text{ and } ghz = hgz = ggz = hhz.$$

Putting $u = p$ and $v = z$ in (2), we get

$$fp = gz = p.$$

Therefore $fp = gp = hp = p$.

Uniqueness of p easily follows from (2).

Theorem 2. *Let $(X, \mathcal{F}, t)$ be a nonarchimedean Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0,1]$ and $(f; g, h)$ an F-type quasi-contraction triplet of self mappings on X. If there is a point u_0 and a sequence $\{u_n\}$ in X such that*

(i) $gu_{2n+1} = fu_{2n}, hu_{2n+2} = fu_{2n+1}, \quad n = 0, 1, 2, \cdots;$

(ii) $g(X) \cap h(X)$ *is $(fu_0; g, h)$-orbitally complete ;*

(iii) f *is $(fu_0; g, h)$-preorbitally commuting with each of g and h, then f, g, h have a unique common fixed point and $\{fu_n\}$ converges to the fixed point.*

Proof. It is easy to see that $\{fu_n\}$ is a Cauchy sequence and in view of (ii), it converges to a point in $g(X) \cap h(X)$. Call it p. Then there exists a point z in X such that $gz = p$. Since $\{fu_n\}$ is a Cauchy sequence, there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$(5) \qquad F_{fu_{2n+1}, fu_{2n+2}}(\varepsilon/k) > 1 - \lambda \text{ and } F_{u, fu_{2n+2}}(\varepsilon/k) > 1 - \lambda$$

for all $n \geq N$..

Now we prove that $fz = gz$.

Taking $u = z$ and $v = u_{2n+2}$ in (3), we have

$$\begin{aligned} F_{fz, fu_{2n+2}}(\varepsilon) \geq\ & \max\{F_{gz, fu_{2n+1}}(\varepsilon/k), F_{fz, gz}(\varepsilon/k), F_{fu_{2n+2}, fu_{2n+1}}(\varepsilon/k), \\ & F_{fz, fu_{2n+1}}(\varepsilon/k), F_{fu_{2n+2}, gz}(\varepsilon/k)\} \\ \geq\ & \max\{F_{u, fu_{2n+2}}(\varepsilon/k), F_{fu_{2n+2}, fu_{2n+1}}(\varepsilon/k), \\ & F_{fz, fu_{2n+2}}(\varepsilon/k), F_{fu_{2n+2}, u}(\varepsilon/k), \\ & F_{fu_{2n+2}, fu_{2n+1}}(\varepsilon/k), F_{fz, fu_{2n+2}}(\varepsilon/k), \\ & F_{fu_{2n+2}, fu_{2n+1}}(\varepsilon/k), F_{fu_{2n+2}, u}(\varepsilon/k)\} \\ >\ & 1 - \lambda, \quad \text{by } (5). \end{aligned}$$

So $fz = gz$ since $fu_{2n+2} \to p$. Similarly for a w in X such that $hw = p$, we can show that $hw = fw = p$. Now by (iii), $fp = fgz = gfz = gu$ and $fu = fhw = hu$ since $gz = u$ and $hw = u$. Also, $fu = fgz = fp$. Therefore $fu = gu = hu$. Putting $u = fu_{2n+1}, v = u$ in (3) and passing to the limits we get $fu = u$. So $fu = gu = hu = u$. Uniqueness follows easily.

Corollary 1. *Let X and f, g, h be as in Theorem 1. If there exists a point u_0 in X and a sequence $\{u_n : n = 0, 1, \cdots\}$ in X such that*

(i) $hu_{2n+1} = fu_{2n}, hu_{2n+2} = gu_{2n+1}, \quad n = 0, 1, 2, \cdots$;

(ii) *$h(X)$ is $(f, g; hu_0)$-orbitally complete ;*

(iii) *h commutes $(f, g; hu_0)$-preorbitally either with f or g ; and*

(iv) *h is $(f, g; hu_0)$-orbitally continuous. Then the conclusions of Theorem 1 hold.*

Extension to Uniform Spaces

Let $D = \{d_\alpha\}$ be a nonempty collection of pseudometrices on X. It is well known that the uniformity generated by D is obtained by taking as a subbase of all sets of the form $U_{\alpha,\varepsilon} = \{(u, v) \in X \times X : d_\alpha(u, v) < \varepsilon\}$, where $d_\alpha \in D$ and $\varepsilon > 0$. In fact, the topology determine by the uniformity has all d_α–spheres as a subbase. For details one may refer to Kelley [9].

Cain and Kasriel [2] have shown that a collection of pseudo–metrics $\{d_\alpha\}$ can be defined which generates the usual structure for Menger spaces. Hence the following result is a direct consequence of Theorem 1.

Theorem 3. *Suppose X is a Hausdorff space and $f, g, h; X \to X$ having the property that for every $d_\alpha \in D$ there is a constant $k_\alpha \in (0, 1)$ such that*

$$(6) \qquad d_\alpha(fu, gv) \leq k_\alpha\{\max d_\alpha(hu, hv), d_\alpha(fu, hu), d_\alpha(gv, hv), d_\alpha(fu, hv), d_\alpha(gv, hu)\};$$

$h(X)$ is sequentially complete ; and h is commuting with f and g, then f, g and h have a unique common fixed point. In an analogous blend Theorem 2 may also be extended to uniform spaces.

Theorem 3 includes a number of fixed point theorems in metric, Menger and uniform spaces, which may be obtained by choosing f, g, h suitably. In particular, if $f = g$ then it presents a nice generalization of Jungck's result [8]. Theorem 4 also includes an interesting result of Khan and Fisher [10] and contains as a special case Theorem 1.1 of Tarafdar [16]. It may be noted that Tarafdar [16] has obtained an exact analogue of Banach contraction mapping principle on a complete Hausdorff space.

Application to Product Spaces

We now give an application of Theorem 2 to product spaces.

Theorem 4. *Let X be a Hausdorff space and $f, g, h : X \times X \to X$. If for every $d_\alpha \in D$, there exists a constant $k_\alpha \in (0,1)$ such that*

$$
\begin{aligned}
d_\alpha(f(u,v), g(u',v')) \leq\ & k_\alpha \max\{d_\alpha(h(u,v), h(u',v')), \\
& d_\alpha(f(u,v), h(u,v)), d_\alpha(g(u',v'), h(u',v')), \\
& d_\alpha((f(u,v), h(u',v')), d_\alpha(g(u',v'), h(u,v))\}
\end{aligned}
\tag{7}
$$

for all $u, v, u', v' \in X$; $h(X \times \{v\})$ is sequentially complete, and

$$
\begin{aligned}
h(f(u,v), v) &= f(h(u,v), v) \\
h(g(u,v), v) &= g(h(u,v), v)
\end{aligned}
$$

for all $u, v \in X$. Then there exists exactly one point $p \in X$ such that

$$f(p,v) = g(p,v) = h(p,v) = p$$

for all $u, v \in X$.

Proof. For a fixed $v \in X$ and $v \neq v'$, the inequality (7) corresponds to (6). Therefore in view of the conclusions of Theorem 2 for each $v \in X$ there exists a unique $u(v)$ in X such that

$$f(u(v), v) = g(u(v), v) = h(u(v), v) = u(v).$$

Now for every $v, v' \in X$ and $d_\alpha \in D$ from (7) we obtain

$$
\begin{aligned}
d_\alpha(u(v), u(v')) &= d_\alpha(f(u(v), v), g(u(v'), v')) \\
&\leq k_\alpha d_\alpha(u(v), u(v')).
\end{aligned}
$$

Consequently $u(v) = u(v')$. So $u(\cdot)$ is some constant $p \in X$ and hence the proof.

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