Kyungpook Mathematical Journal
Volume 31, Number 1, June, 1991

ON THE WEAK AUTOMORPHISM GROUP OF A PRINCIPAL BUNDLE, PRODUCT CASE

Chan-Young Park

In this paper, we will establish a homotopy theory of the weak automorphism group of a principal bundle in the product case, which serves as the universal group in the Seifert fiber space construction. (See [LR])

1. Preliminaries

Let G be a Lie group and let $q : P \to B$ be the projection map of a principal G-bundle. Assume that the base space B is locally compact, locally connected Hausdorff space. Then by R. Palais ([P_1]) the projection map $q : P \to B$ is equivalent to the quotient map of a proper free left G-action on P.

By $Top(B)$ we denote the topological group of all self-homeomorphisms of B, equipped with the compact-open topology. By $Aut(G)$ we denote the Lie group of all continuous automorphisms of G. A self-homeomorphism f of P is called a *weak automorphism* of the principal G-bundle if there exists a continuous automorphism $a \in Aut(G)$ so that $f(gx) = a(g)f(x)$ for all $g \in G$ and $x \in P$. When a is the identity automorphism of G, we call f an *automorphism*. By $Aut_G(P)$ and respectively $Aut_G^w(P)$ we denote the topological group of all the automorphisms and all the weak automorphisms, respectively, of the principal bundle.

Thus we have a natural projection $p : Aut_G^w(P) \longrightarrow Top(B)$ via $p(f) =$

Received December 12, 1990.
Partially supported by a grant from KOSEF.

$\overline{f}$ where $\overline{f}$ is induced from f, making the following diagram commute :

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\overline{f}]{} & B. \end{array}$$

Clearly the map p is a homomorphism.

Theorem 1.1. *The projection map $p : Aut^w_G(P) \to Top(B)$ is continuous.*

Proof. Let $q : P \to B$ denote the projection map of the principal G–bundle. Let $f \in Aut^w_G(P)$ with $p(f) = h$. Assume that h is contained in a basic open set (C, U), i.e., C and U are compact and open, respectively, subsets of B and $h(C) \subset U$. Since $C \subset h^{-1}(U)$ and B is locally compact, C is covered by finitely many open subsets U_α of B such that $\overline{U}_\alpha$ is compact, $h(\overline{U}_\alpha) \subset U$ and $q^{-1}(\overline{U}_\alpha)$ is trivial for each α. Then C is a finite union of compact subsets of the form $C \cap \overline{U}_\alpha$ and so (C, U) contains a finite intersection of the open subsets of the form $(C \cap \overline{U}_\alpha, U)$. Since $(C \cap \overline{U}_\alpha, U)$ contains h and $q^{-1}(C \cap \overline{U}_\alpha)$ is trivial, we may assume without loss of generality that (C, U) is such that $q^{-1}(C)$ and $q^{-1}(U)$ are trivial. Now identify $q^{-1}(C)$ with $G \times C$ and $q^{-1}(U)$ with $G \times U$. Then $(\{e\} \times C, G \times U) \cap Aut^w_G(P)$ is an open subset of $Aut^w_G(P)$ and $f \in (\{e\} \times C, G \times U)$ and $(\{e\} \times C, G \times U) \cap Aut^w_G(P)$ is contained in $p^{-1}(C, U)$ where e denotes the identity element of G. Thus the map p is continuous.

Now what is the kernal of p ? It is known in [LR] that the kernal of p is precisely $Map^w_G(P, G)$ the topological group of all continuous maps $\lambda \in Map(P, G)$ satisfying

$$\lambda(gx) = a(g)\lambda(x)g^{-1}$$

for all $g \in G, x \in P$ and for some $a \in Aut(G)$ where $Map(P, G)$ denotes the space of all continuous maps from P to G. The topology of $Map^w_G(P, G)$ is of course the compact–open topology and the group operation $*$ on $Map^w_G(P, G)$ is defined as follows : for $\lambda_1, \lambda_2 \in Map^w_G(P, G)$ with $\lambda_i(gx) = a_i(g)\lambda_i(x)g^{-1}$, $\lambda_2 * \lambda_1 : P \to G$ is defined by $\lambda_2 * \lambda_1(x) = a_2(\lambda_1(x))\lambda_2(x)$ for all $x \in P$.

Let $Map_G(P, G) = \{\lambda \in Map(P, G) \mid \lambda(gx) = g\lambda(x)g^{-1}$ for all $g \in G$ and $x \in P\}$.

Summarizing we have the following

Theorem 1.2. *For a principal G–bundle as above, we have a short exact sequence of topological groups:*

$$1 \longrightarrow Map^w_G(P,G) \underset{i}{\longrightarrow} Aut^w_G(P) \longrightarrow Im(p) \longrightarrow 1$$

where $Im(p)$ denotes the image of the projection map $p : Aut^w_G(P) \to Top(B)$ and the map $i : Map^w_G(P,G) \to Aut^w_G(P)$ is defined by $i(\lambda)(x) = \lambda(x)x$ for all $x \in P$.

Theorem 1.3. ([CR$_2$], [LR]) *Let P be a trivial principal G-bundle over B. Then we have a splitting exact sequence of topological groups:*

$$1 \longrightarrow Map_G(P,G) \longrightarrow Aut^w_G(P) \longrightarrow Aut(G) \times Top(B) \longrightarrow 1.$$

Proof. A stronger version of this theorem is stated in [CR$_2$] for $G = \mathbf{R}^k$ and in [LR] for general product bundle. Actually $Aut^w_G(P)$ is a semi-direct product of $Map_G(P,G)$ and $Aut(G) \times Top(B)$. We outline the proof. Let $f \in Aut^w_G(P)$. Then there exists a unique continuous automorphism $\alpha \in Aut(G)$ such that $f(gx) = \alpha(g)f(x)$ for all $g \in G$ and $x \in P$. Also f induces a homeomorphism $\overline{f} \in Top(B)$ as above. Define a map p' : $Auto^w_G(P) \longrightarrow Aut(G) \times Top(B)$ by $p'(f) = (\alpha, \overline{f})$. Then p' is a continuous homomorphism and the kernel of p' is precisely the space $Map_G(P,G)$. Now conversely for $(\beta, h) \in Aut(G) \times Top(B)$, define $\widetilde{h} : P \to P$ by $\widetilde{h}(g,x) = (\beta(g), h(x))$ for $(g,x) \in P = G \times B$. Then $\widetilde{h} \in Aut^w_G(P)$ and thus defines a map $s : Aut(G) \times Top(B) \to Aut^w_G(P)$ via $s(\beta,h) = \widetilde{h}$. Clearly s is a continuous homomorphism and $p \circ s =$ the identity map on $Aut(G) \times Top(B)$. This completes the proof.

Corollary 1.4. *Let P be a trivial principal G-bundle over B. Then we have splitting exact sequences of topological groups:*

$$1 \longrightarrow Map^w_G(P,G) \longrightarrow Aut^w_G(P) \longrightarrow Top(B) \longrightarrow 1$$

$$1 \longrightarrow Map_G(P,G) \longrightarrow Aut_G(P) \longrightarrow Top(B) \longrightarrow 1$$

and

$$1 \longrightarrow Map_G(P,G) \longrightarrow Map^w_G(P,G) \longrightarrow Aut(G) \longrightarrow 1.$$

2. Fibrations

Let $q : P \to B$ be the projection map of a principal G–bundle as in §1.

Lemma 2.1. *Assume that $B = Y \times I$ and that every open subspace of Y is paracompact. Then there exists a principal G–bundle E over Y and a G–bundle equivalence $\varphi : P \to E \times I$ of principal G–bundles such that*

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & E \times I \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ B & = & Y \times I \end{array}$$

commutes where G acts on I trivially and q' is the projection map. Moreover, E can be taken to be $q^{-1}(Y \times \{0\})$ and

$$\varphi|_{q^{-1}(Y \times \{0\})} : E \to E \times I,$$

the inclusion $x \mapsto (x, 0)$ for all $x \in E$.

Proof. It is very similar to that of theorem 7.1 of Bredon ([B] p 93).

Theorem 2.2. (Weak Covering Homotopy Theorem) *Let X_1, X_2 be two principal G–bundles over base spaces B_1 and B_2 respectively. Assume that every open subspace of B_1 is paracompact. Let $f : X_1 \to X_2$ be a weakly G–equivariant map, say $f(gx) = a(g)f(x)$ for some $a \in Aut(G)$. Let $h : B_1 \to B_2$ be the map induced from f and let $H : B_1 \times I \to B_2$ be a homotopy starting at h. Then there exists a weakly G–equivariant homotopy $F : X_1 \times I \to X_2$ covering H and starting at f. Furthermore $F(gx, t) = a(g)F(x, t)$ for all $g \in G$, $x \in X_1$ and $t \in I$.*

Proof. Consider the pull-back $h^*(X_2)$ over B_1 of the map $h : B_1 \to B_2$. Recall that $h^*(X_2) = \{(x^*, y) \in B_1 \times X_2 | h(x^*) = y^*\}$ where x^*, y^* are the projecitons of $x \in X_1$ and $y \in X_2$ respectively. By the universal property of the pull–backs, there exists a unique map $k : X_1 \to h^*(X_2)$ such that the diagram

$$
\begin{array}{ccccc}
X_1 & & \xrightarrow{f} & & X_2 \\
 & \overset{k}{\searrow} & & \nearrow_{\widetilde{h}} & \\
 & & h^*(X_2) & & \\
{\scriptstyle q_1}\downarrow & & & & \downarrow{\scriptstyle q_2} \\
 & \swarrow_{h^*(q_2)} & & & \\
B_1 & & \underset{h}{\longrightarrow} & & B_2
\end{array}
$$

commutes. In fact $k(x) = (x^*, f(x)),\ \ x \in X$. The identities $k(gx) = (x^*, f(gx)) = a(g)(x^*, f(x)) = a(g)k(x)$ imply that k is actually a weakly G-equivariant map. Consider the pull–back $H^*(X_2)$ over $B_1 \times I$:

$$
\begin{array}{ccc}
H^*(X_2) & \xrightarrow{\widetilde{H}} & X_2 \\
{\scriptstyle H^*(q_2)}\downarrow & & \downarrow{\scriptstyle q_2} \\
B_1 \times I & \underset{H}{\longrightarrow} & B_2
\end{array}
$$

Then $H^*(q_2)^{-1}(B_1 \times \{0\}) = h^*(X_2)$ and by lemma 2.1, there exists a G–equivalence $\varphi : h^*(X_2) \times I \to H^*(X_2)$ such that $\varphi((x^*, y), 0) = (x^*, y)$ for all $(x^*, y) \in h^*(X_2)$. Define $F : X_1 \times I \to X_2$ by

$$F(x,t) = \widetilde{H} \circ \varphi(k(x), t), \quad x \in X_1, \quad t \in I.$$

Then $F(g(x,t)) = F(gx,t) = \widetilde{H} \circ \varphi(k(gx),t) = \widetilde{H} \circ \varphi(a(g)k(x),t) = \widetilde{H}(a(g)\varphi(k(x),t)) = a(g)F(x,t)$ for all $g \in G$, $x \in X_1$ and $t \in I$ and $F(x,0) = \widetilde{H} \circ \varphi(k(x),0) = \widetilde{h}(k(x)) = f(x)$. Thus F is the desired homotopy.

Theorem 2.3. *Let P be a principal G–bundle over B. Assume that every open subspace of B is paracompact. Then the projection map*

$$p : Aut^w_G(P) \longrightarrow Top(B)$$

is a Serre fibration.

Proof. Note that B is a locally compact, locally connected Hausdorff space and recall that a Serre fibration is a map satisfying the homotopy lifting property for any CW–complex. Let Y be a CW–complex and suppose that we have a map $f : Y \to Aut^w_G(P)$ and a homotopy $h_t : Y \to Top(B)$, $t \in I$ such that $p \circ f = h_0$. Consider the principal G–bundle $P \times Y \to B \times Y$ where G acts on Y trivially. The maps $f : Y \to Aut^w_G(P)$ and $h_t : Y \to$

$Top(B)$ induce maps $f' : P \times Y \to P \times Y$ and $h'_t : B \times Y \to B \times Y$ respectively via the equations :

$$\begin{aligned} f'(x,y) &= (f(y)(x),y) \quad \text{and} \\ h'_t(b,y) &= (h_t(y)(b),y) \end{aligned}$$

for all $x \in P$, $y \in Y$ and $b \in B$.

Clearly f' and h'_t are homeomorphisms for each $t \in I$ so that h'_t is an isotopy.
Furthermore if $f(y)(gx) = a_y(g)f(y)(x)$ for $a_y \in Aut(G)$, then $f'(g(x, y)) = f'(gx,y) = (f(y)(gx),y) = a_y(g)f'(x,y)$. Since every subset of a CW–complex is paracompact, every open subset of $B \times Y$ is paracompact. Thus by applying theorem 2.2, we have a homotopy $f'_t : P \times Y \to P' \times Y$ starting at f', covering h'_t and such that $f'_t(g(x,y)) = a_y(g)f'_t(x,y)$ for $g \in G$ and $(x,y) \in P \times Y$. By using the equation : $f'_t(x,y) = (f_t(y)(x),y)$ for all $(x,y) \in P \times Y$, f'_t induces a homotopy $f_t : Y \to Top(P)$ covering h_t and starting at f. Since f'_t fixes the second factor Y,

$$(f_t(y))(gx) = a_y(g)(f_t(y))(x)$$

for all $y \in Y$, $g \in G$ and $x \in P$. This implies $f_t \in Aut^w_G(P)$ and completes the proof.

Corollary 2.4. *Under the same assumption as in theorem 2.3, the projection map $Aut_G(P) \to Top(B)$ is a Serre fibration.*

Remark 2.5. For two principal G–bundles $P_1 \to B_1$ and $P_2 \to B_2$, we can define the space of all weakly G–equivariant maps $Map^w_G(P_1,P_2)$ as well as $Map_G(P_1,P_2)$, i.e., a map $f \in MaP(P_1,P_2)$ belongs to $Map^w_G(P_1,P_2)$ if and only if there exists a continuous automorphism $a \in Aut(G)$ such that $f(gx) = a(g)f(x)$ for all $g \in G$ and $x \in P_1$. Also we can define the natural projections

$$p : Map^w_G(P_1,P_2) \longrightarrow Map(B_1,B_2)$$

and

$$p' : Map_G(P_1,P_2) \longrightarrow Map(B_1,B_2).$$

By using similar arguments employed as in theorems 2.2 and 2.3, we can show that the above natural projections are Serre fibrations. In [J], I. M. James showed that the second projection map $p' : Map_G(P_1,P_2) \to$

$Map(B_1, B_2)$ is a Serre fibration provided that B_1 is a locally finite CW–complex and G is a compact topological group.

3. Some Calculations

In this section, we will restrict ourselves to the case when G is a k–torus T^k and B is a compact connected manifold so that P is a principal T^k–bundle over a compact connected manifold B.

Let $Map_c(B, T^k)$ be the subgroup of the topological group $Map(B, T^k)$ which are homotopic to a constant mapping.

Theorem 3.1. *If B is a compact connected topological space, then*

$$\pi_i Map(B, T^k) = \begin{cases} H^1(B; \mathbf{Z}^k), & i = 0 \\ \mathbf{Z}^k, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The exact sequence $0 \to \mathbf{Z}^k \to \mathbf{R}^k \to T^k \to 0$ induces an exact sequence of topological groups:

$$0 \longrightarrow Map(B, \mathbf{Z}^k) \longrightarrow Map(B, \mathbf{R}^k) \longrightarrow Map_c(B, T^k) \to 0.$$

Moreover, for each $f \in Map_c(B, T^k)$ there exists a unique map $\tilde{f} \in Map(B, \mathbf{R}^k)$ up to an element of $Map(B, \mathbf{Z}^k)$. In particular if f is close to the mapping $c_1 : B \to T^k$, $c_1(b) = 1$ for all $b \in B$, then $\tilde{f}$ can be chosen close to $c_0 : B \to \mathbf{R}^k$, $c_0(b) = 0$ for all $b \in B$. Therefore the exact sequence of topological groups is a locally trivial fiber bundle. Now we have

$$\begin{aligned} \pi_i Map(B, \mathbf{R}^k) &= 0 \quad \text{for all } i \quad \text{and} \\ \pi_i Map(B, \mathbf{Z}^k) &= \pi_i \mathbf{Z}^k = \begin{cases} \mathbf{Z}^k, & i = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Applying the homotopy exact sequence, we obtain

$$0 \longrightarrow \pi_1(Map(B, T^k), c_1) \longrightarrow \pi_0 Map(B, \mathbf{Z}^k) \longrightarrow 0$$

with the remaining groups all 0. Thus

$$\pi_i(Map(B, T^k), c_1) = \begin{cases} \mathbf{Z}^k, & i = 1 \\ 0, & i > 1. \end{cases}$$

Finally in dimension 0, the path–components of $Map(B,T^k)$ are precisely the homotopy classes $[B,T^k]$ which is naturally identified with $H^1(B;\mathbf{Z}^k)$.

Let P be a principal T^k-bundle over a compact connected manifold B. For $f \in Aut^w_{T^k}(P)$, there exists a unique continuous automorphism $a \in Aut(T^k)$ such that $f(gx) = a(g)f(x)$ for all $g \in T^k$ and $x \in P$. This gives us a continuous map

$$p_1 : \ Aut^w_{T^k}(P) \longrightarrow Aut(T^k)$$

defined by $p_1(f) = a$. Thus $Aut^w_{T^k}(P)$ is a principal $Aut_{T^k}(P)$–bundle over the image of p_1 because $Aut(T^k)$ is discrete.

Corollary 3.2. *Let P be a principal T^k–bundle over a compact connected manifold B. If $p_1 : Aut^w_{T^k}(P) \to Aut(T^k)$ is onto, then*

$$\pi_i Map^w_{T^k}(P,T^k) = \begin{cases} \mathbf{Z}^k, & i = 1 \\ 0, & i > 1 \end{cases}$$

and

$$0 \longrightarrow H^1(P;\mathbf{Z}^k) \longrightarrow \pi_0 Map^w_{T^k}(B,T^k) \longrightarrow GL(k;\mathbf{Z}) \longrightarrow 1$$

is exact.

Proof. This follows from the application of the homotopy exact sequence to the locally trivial bundle

$$Map_{T^k}(P,T^k) \longrightarrow Map^w_{T^k}(P,T^k) \longrightarrow Aut(T^k)$$

and using theorem 3.1 together with the facts:

$$Aut(T^k) = GL(k,\mathbf{Z}) \quad \text{and}$$

$$Map_{T^k}(P,T^k) = Map(B,T^k).$$

Remark 3.3. If P is any trivial G–bundle over B, for example when $G = \mathbf{R}^k$ and see [CR$_1$] for many cases when $G = T^k$, then the natural projection maps $p : Aut^w_G(P) \to Top(B)$ and $p_1 : Aut^w_G(P) \to Aut(T^k)$ are onto. When the projection map $p : Aut^w_G(P) \to Top(B)$ is onto, it is possible to relate the homotopy groups of $Aut_G^w(P)$ with those of $Top(B)$. The ontoness of the projection map p in other cases will be studied later.

Example 3.4. Fix $G = T^k (k \geq 1)$ and let P be a trivial principal G–bundle over S^2. Then by theorem 1.3 and corollary 1.4, we have splitting exact sequences of topological groups:

$$1 \longrightarrow Map_G^w(P,G) \longrightarrow Aut_G^w(P) \longrightarrow Top(S^2) \longrightarrow 1,$$

$$1 \longrightarrow Map_G(P,G) \longrightarrow Aut_G(P) \longrightarrow Top(S^2) \longrightarrow 1,$$

$$1 \longrightarrow Map_G(P,G) \longrightarrow Map_G^w(P,G) \longrightarrow Aut(G) \longrightarrow 1.$$

Also recall that $Map_G(P,G)$ is homeomorphic to $Map(S^2,G)$. Thus for all i

$$\begin{aligned} \pi_i Aut_G^w(P) &= \pi_i Map_G^w(P,G) \oplus \pi_i Top(S^2) \\ &= \pi_i Map_G(P,G) \oplus \pi_i Aut(G) \oplus \pi_i Top(S^2) \\ &= \pi_i Map(S^2,G) \oplus \pi_i Aut(G) \oplus \pi_i Top(S^2) \end{aligned}$$

and

$$\begin{aligned} \pi_i Aut_G(P) &= \pi_i Map_G(P,G) \oplus \pi_i Top(S^2) \\ &= \pi_i Map(S^2,G) \oplus \pi_i Top(S^2). \end{aligned}$$

Recall that

$$\begin{aligned} \pi_i Top(S^2) &= \pi_i(O(3)) \\ &= \begin{cases} \mathbf{Z}_2, & i = 0,1 \\ 0, & i = 2 \\ \pi_i(S^3) = \pi_i(S^2), & i > 2 \end{cases} \end{aligned}$$

So we have

$$\pi_i Aut_G^w(P) = \begin{cases} GL(k,\mathbf{Z}) \oplus \mathbf{Z}_2 & i = 0 \\ \mathbf{Z}^k \oplus \mathbf{Z}_2, & i = 1 \\ 0, & i = 2 \\ \pi_i(S^2), & i \geq 3 \end{cases}$$

and

$$\pi_i Aut_G(P) = \begin{cases} \mathbf{Z}_2, & i = 0 \\ \mathbf{Z}^k \oplus \mathbf{Z}_2, & i = 1 \\ 0, & i = 2 \\ \pi_i(S^2), & i \geq 3. \end{cases}$$

References

[B] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press (1972).

[CR_1] P. E. Conner and F. A. Raymond, *Injective operations of the toral groups*, Topology 10 (1971), 283-296.

[CR_2] P. E. Conner and F. A. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifold*, Bull. Amer. Math. Soc. 83(1977), 36-85.

[J] I. M. James, *The space of bundle maps*, Topology 2 (1963), 45–89.

[LR] K. B. Lee and F. A. Raymond, *Seifert manifolds modelled on principal bundles*, to appear.

[P_1] R. S. Palais, *On the existence of slices for non-compact Lie groups*, Annals of Math. II Series 73(1961), 295–323.

[P_2] C. Y. Park, *Homotopy groups of automorphism groups of some Seifert fiber spaces*, Ph.D. Thesis, The University of Michigan (1989).

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702–701, Korea