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# Bayesian Estimation for the Weibull Model under the Progressively Censoring Scheme

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## ABSTRACT

The maximum likelihood estimators and Bayes estimators of the parameters and reliability function for the two-parameter Weibull distribution under the type-II progressively censoring schemes are derived when a shape parameter is known and unknown, respectively. Efficiencies for above estimators are also compared each other in terms of the mean square errors, and Bayes risk sensitivities of the Bayes estimators are investigated.

## 1. Introduction

The two-parameter Weibull distribution has been widely used in the field of the reliability and life testing.

The probability density function (p.d.f.) of the two-parameter Weibull distribution is given by

$$f(x \mid \theta,\gamma) \;=\; \frac{\gamma}{\theta}x^{\gamma-1}\exp(-\frac{x^{\gamma}}{\theta}),\;\; 0 \;< x \;< \infty,\; \theta,\gamma \;>\; 0, \tag{1.1}$$

where $\theta$ and $\gamma$ are referred to as scale and shape parameters, respectively, and denoted by $\mathcal{W}(\theta,\gamma)$.

Let the reliability function denote the probability of survival until the mission time $t_0$. Then the reliability function $R(t_0 \mid \theta,\gamma)$ of $\mathcal{W}(\theta,\gamma)$ is

$$R(t_0 \mid \theta,\gamma) \;=\; \exp(-t_0^{\gamma}/\theta),\;\; 0 \;<\; t_0 \;< \infty,\;\; \theta,\gamma \;>\; 0. \tag{1.2}$$

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Hart and Moore(1965) obtained the maximum likelihood estimator(M.L.E.) of a scale parameter in the two-parameter Weibull distribution under the failure-censored case. The M.L.E. and uniformly minimum variance unbiased estimator of the reliability function for the two-parameter Weibull distribution with a known shape parameter have been proposed by Basu(1964). Cohen(1965) considered the M.L.E.'s of both scale and shape parameters. In the Bayesian estimation of the parameters and reliability function for the two-parameter Weibull distribution in the case where a scale parameter is a random variable, Soland(1968) considered the gamma prior distribution and Canovos and Tsokos(1973) used the exponential, inverted gamma and uniform prior distributions for a scale parameter. In the case where both scale and shape parameters are random variables, the Bayes estimators of the parameters and reliability function for $\mathcal{W}(\theta, \gamma)$ have been considered by Bury(1972), Canovos and Tsokos(1973) and Papadopulos and Tsokos(1975). Progressively censoring schemes are often used in clinical trials and life testing problems with a view to monitoring the experiment from the start with the objective of a possible early termination of the experiment depending on the cumulative at its various steps. For the progressively censoring scheme, Cohen(1965) studied M.L.E.'s of both scale and shape parameters, Gibbson and Vanse(1983) obtained M.L.E. and least squares median ranks estimator, and Caciari and Montanari(1987) considered the confidence limits for parameters.

In this paper, the M.L.E.'s and Bayes estimators of the parameters and reliability function for the two-parameter Weibull distribution under the type-II progressively censoring schemes are derived when a shape parameter is known and unknown, respectively. Efficiencies for above estimators are also compared each other in terms of the mean square errors(M.S.E.), and the Bayes risk sensitivities of the Bayes estimators are investigated.

In Section 2, for the case of a shape parameter known, we derive the M.L.E. of the parameter and reliability function, the generalized maximum likelihood estimator(G.M.L.E.), and the Bayes estimators of the parameter and reliability function derived under the noninformative and inverted gamma prior distribution. Also, for the case of a shape parameter unknown, the M.L.E. of the parameters and reliability function are obtained and the Bayes estimators of the parameters and reliability function are derived under the inverted gamma and uniform prior distribution for a scale parameter, while the independent uniform prior distribution for a shape parameter.

In Section 3, for the case of a shape parameter known, we obtain the Bayes risk of the Bayes estimator when the true prior distribution is not the inverted gamma prior distribution .

In Section 4, through the Monte Calro simulation study, we compare the Bayes estimators with the M.L.E.'s in terms of the M.S.E.'s for a shape parameter known and unknown, respectively.

## 2. Estimation of Parameters and Reliability Function under the Type-II Progressively Censored Case

Let $N$ denote the total sample size and $n$ the number of sample specimens which result in completely determined life spans. Suppose that censoring occurs progressively in k-stage at time $T_i$ such that $T_i > T_{i-1}, i = 1, \cdots, k$, and that at the $i$th stages of censoring $r_i$ sample specimens selected randomly from the survivors at time $T_i$ are removed(censored) from further observation. Therefore, it follows that

$$N = n + \sum_{i=1}^{k} r_i.$$

The likelihood function under the type-II progressively censored sample is given by

$$\begin{aligned} L = l(\theta, \gamma \mid x) &= \prod_{i=1}^{n} (n_i^* f(x_i)[1 - F(x_i)]^{r_i}) \\ &= c(\gamma/\theta)^n (\prod_{i=1}^{n} x_i^{\gamma-1}) \exp\left[ -\sum_{i=1}^{n} (1 + r_i) x_i^{\gamma} \right], \end{aligned} \tag{2.1}$$

where $n_i^* = N - \sum_{j=1}^{i-1} r_j - i + 1$.

We use the squared error loss functions for $\theta$, $\gamma$ and the reliability function $R(t_0 \mid \theta, \gamma)$ given by as follows;

$$L(\theta,\ \theta^*) = (\theta - \theta^*)^2, \tag{2.2}$$

$$L(\gamma,\ \gamma^*) = (\gamma - \gamma^*)^2, \tag{2.3}$$

and

$$L(R(t_0 \mid \theta,\gamma),\ R^*(t_0 \mid \theta,\gamma)) \ = \ (R(t_0 \mid \theta,\gamma) \ - \ R^*(t_0 \mid \theta,\gamma))^2, \qquad (2.4)$$

where $\theta^*$, $\gamma^*$ and $R^*(t_0 \mid \theta,\gamma)$ are the estimators of $\theta$, $\gamma$ and $R(t_0 \mid \theta,\gamma)$, respectively.

**Theorem 1.** When $\gamma$ is known, the M.L.E.'s of $\theta$ and the reliability function $R(t_0 \mid \theta,\gamma)$ under the type-II progressively censoring scheme are given by

$$\widehat{\theta} \ = \ \frac{1}{n}\sum_{i=1}^{n}(1+r_i)x_i^{\gamma} \ , \qquad (2.5)$$

and

$$\widehat{R(t_0)} \ = \ \exp(-t_0^{\gamma}/\widehat{\theta}), \qquad (2.6)$$

respectively.

Now, we consider the Bayes estimators and G.M.L.E.'s for the parameter and reliability function.

**Theorem 2.** When $\gamma$ is known, the Bayes estimators of the parameter and reliability function under the type-II progressively censored case are as follows;
(i) For the noninformative prior distribution for $\theta$,

$$\theta_1^* \ = \ \frac{S_n}{n-1} \ , \qquad (2.7)$$

and

$$R_1^* \ = \ \frac{1}{(1+t_0^{\gamma}/S_n)^n} \ . \qquad (2.8)$$

(ii) For the inverted gamma prior distribution for $\theta$,

$$\theta_2^* \ = \ \frac{S_n+\mu}{n+\nu-1} \ , \qquad (2.9)$$

and

$$R_2^* \ = \ \frac{1}{(1+t_0^{\gamma}/(S_n+\mu))^{n+\nu}} \ , \qquad (2.10)$$

where,

$$S_n = \sum_{i=1}^{n}(1+r_i)x_i^\gamma .$$

**Proof.** (i) The noninformative prior distribution for $\theta$ is

$$g_1(\theta) \propto \frac{1}{\theta} , \quad \theta > 0 ,$$

and the joint distribution of $\theta$ and $\underline{x} = (x_1, \cdots, x_n)$ is

$$h(\theta, \underline{x}) = c\gamma^n (\prod_{i=1}^{n} x_i^{\gamma-1})(\exp - \frac{S_n}{\theta})/\theta^{n+1} .$$

Thus, the posterior distribution of $\theta$ is

$$\pi(\theta \mid \underline{x}) = \frac{h(\theta, \underline{x})}{\int_0^\infty h(\theta, \underline{x}) d\theta} = \frac{S_n^n \exp(-S_n/\theta)}{\Gamma(n)\theta^{n+1}} . \tag{2.11}$$

Therefore, under the squared error loss functions (2.2) and (2.4), the Bayes estimator $\theta_1^*$ of $\theta$ is

$$\begin{aligned}
\theta_1^* &= \int_0^\infty \theta \pi(\theta \mid \underline{x}) d\theta \\
&= \int_0^\infty \theta \frac{S_n^n \exp(-S_n/\theta)}{\Gamma(n)\theta^{n+1}} d\theta \\
&= \frac{S_n}{n-1} ,
\end{aligned}$$

and the Bayes estimator $R_1^*$ of $R(t_0 \mid \theta, \gamma)$ is

$$\begin{aligned}
R_1^* &= E\left[\exp(-t_0^\gamma/\theta) \mid \underline{x}\right] \\
&= \int_0^\infty \exp(-t_0^\gamma/\theta) \pi(\theta \mid \underline{x}) d\theta \\
&= \int_0^\infty \exp(-t_0^\gamma/\theta) \frac{S_n^n \exp(-S_n/\theta)}{\Gamma(n)\theta^{n+1}} d\theta \\
&= \frac{1}{(1 + t_0^\gamma/S_n)^n} .
\end{aligned}$$

(ii) Similarly, we can prove (ii).

**Corollary.** When $\gamma$ is known, the G.M.L.E.'s of the parameter and reliability function under the type-II progressively censored case are given by as follows;
(i) For the noninformative prior distribution for $\theta$,

$$\theta_3^* = \frac{S_n}{n+1} ,$$

and

$$R_3^* = \exp(-t_0^\gamma/\theta_3^*) .$$

(ii) For the inverted gamma prior distribution for $\theta$,

$$\theta_4^* = \frac{S_n+\mu}{n+\nu+1} ,$$

and

$$R_4^* = \exp(-t_0^\gamma/\theta_4^*) .$$

**Theorem 3.** When $\gamma$ is unknown, the M.L.E.'s of the parameters and reliability function under the type-II progressively censored case become the solutions $\tilde{\theta}$ and $\tilde{\gamma}$ of the equations

$$\left[\frac{\sum_{i=1}^n(1+r_i)x_i^{\tilde{\gamma}}\ln x_i}{\sum_{i=1}^n(1+r_i)x_i^{\tilde{\gamma}}} - \frac{1}{\tilde{\gamma}}\right] = \frac{1}{n}\sum_{i=1}^n \ln x_i ,$$

$$\tilde{\theta} = \frac{1}{n}\sum_{i=1}^n(1+r_i)x_i^{\tilde{\gamma}} ,$$

and

$$\widetilde{R(t_0)} = \exp(-t_0^{\tilde{\gamma}}/\tilde{\theta}) ,$$

respectively.

Now, we consider the Bayes estimators for the parameters and reliabilty function.

**Theorem 4.** When $\gamma$ is unknown, the Bayes estimators of the parameters and reliability function under the type-II progressively censored case are obtained by as follows;

(i) For the inverted gamma prior distribution for $\theta$ and independent uniform prior distribution for $\gamma$,

$$\theta_1^{**} = \frac{\int_a^b \gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu)^{n+\nu-1} d\gamma}{(n+\nu-1)\int_a^b \gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu)^{n+\nu} d\gamma},$$

$$\gamma_1^{**} = \frac{\int_a^b \gamma^{n+1} \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu)^{n+\nu} d\gamma}{\int_a^b \gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu)^{n+\nu} d\gamma},$$

and

$$R_1^{**} = \frac{\int_a^b \gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu + t_0^\gamma)^{n+\nu} d\gamma}{\int_a^b \gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + \mu)^{n+\nu} d\gamma},$$

where, $0 < \theta < \infty, \quad a < \gamma < b$.

(ii) For both the uniform prior distribution for $\theta$ and $\gamma$,

$$\theta_2^{**} = \frac{\int_a^b (\gamma^n \prod_{i=1}^n x_i^{\gamma-1} / S_n^{n-2}) \Gamma^c(n-2, S_n/\delta) d\gamma}{\int_a^b (\gamma^n \prod_{i=1}^n x_i^{\gamma-1} / S_n^{n-1}) \Gamma^c(n-1, S_n/\delta) d\gamma},$$

$$\gamma_2^{**} = \frac{\int_a^b (\gamma^{n+1} \prod_{i=1}^n x_i^{\gamma-1} / S_n^{n-1}) \Gamma^c(n-1, S_n/\delta) d\gamma}{\int_a^b (\gamma^n \prod_{i=1}^n x_i^{\gamma-1} / S_n^{n-1}) \Gamma^c(n-1, S_n/\delta) d\gamma},$$

and

$$R_2^{**} = \frac{\int_a^b (\gamma^n \prod_{i=1}^n x_i^{\gamma-1} / (S_n + t_0^\gamma)^{n-1}) \Gamma^c(n-1, (S_n + t_0^\gamma)/\delta) d\gamma}{\int_a^b (\gamma^n \prod_{i=1}^n x_i^{\gamma-1} / S_n^{n-1}) \Gamma^c(n-1, S_n/\delta) d\gamma},$$

$$0 < \theta < \delta, \quad a < \gamma < b,$$

where, $\Gamma^c(a, c)$ is the complement of the incomplete gamma function defined by

$$\Gamma^c(a, c) = \int_x^\infty y^{a-1} \exp(-y) dy = \Gamma(a) - \Gamma(a, x).$$

## 3. Robustness of the Bayes Estimator

An important consideration in the Bayesian analysis concerns the sensitivity of the performance of the Bayes estimator derived under the assumed prior distribution.

When the assumed prior distribution is the inverted gamma distribution and the true prior distribution is not the inverted gamma prior distribution, we obtain the sensitivity for the Bayes estimator of $\theta$.

**Theorem 5.** The Bayes risk's of the Bayes estimator and M.L.E. of $\theta$ follow as;

(i) When the true prior distribution is the uniform distribution, the Bayes risk of the the Bayes estimator $\theta_2^*$ of $\theta$ is

$$r(\mathcal{U}(0,b),\ \theta_2^*) = \frac{b^2}{3(n+\nu-1)^2}T_1 + \frac{b\mu}{(n+\nu-1)^2}T_2 + \frac{\mu^2}{(n+\nu-1)^2},$$

and that of the M.L.E. $\widehat{\theta}$ is

$$r(\mathcal{U}(0,b),\ \widehat{\theta}) = \frac{b^2}{3n^2}T_3 .$$

(ii) When the true prior distribution is the gamma distribution, the Bayes risk of the Bayes estimator $\theta_2^*$ of $\theta$ is

$$r(g(\alpha,\beta),\ \theta_2^*) = \frac{\alpha(\alpha+1)}{(n+\nu-1)^2\beta^2}T_1 + \frac{2\mu\alpha}{(n+\nu-1)^2\beta}T_2 + \frac{\mu^2}{(n+\nu-1)^2},$$

and that of the M.L.E. of $\theta$ is

$$r(g(\alpha,\beta),\ \widehat{\theta}) = \frac{\alpha(\alpha+1)}{n^2\beta^2}T_3 .$$

where

$$T_1 = (n+\nu-1)^2 - 2(n+\nu-1)\sum_{i=1}^{n}(1+r_i) + \sum_{i=1}^{n}(1+r_i)^2 + \left(\sum_{i=1}^{n}(1+r_i)\right)^2,$$

$$T_2 = \sum_{i=1}^{n}(1+r_i) - (n+\nu-1),$$

$$T_3 = n^2 - 2n\sum_{i=1}^{n}(1+r_i) + \left(\sum_{i=1}^{n}(1+r_i)\right)^2 + \sum_{i=1}^{n}(1+r_i)^2 .$$

**Proof.** The Bayes risk of the Bayes estimator $\theta_2^*$ of $\theta$ is

$$\begin{aligned} r(\mathcal{U}(0,b),\ \theta_2^*) &= E^{\theta}[E^{\underline{x}}(\theta - \theta_2^* \mid \theta)^2] \\ &= E^{\theta}[E^{\underline{x}}(\theta^2\ -\ 2\theta\theta_2^*\ +\ \theta_2^{*2} \mid \theta)] \\ &= \frac{b^2}{3(n+\nu-1)^2}T_1\ +\ \frac{b\mu}{(n+\nu-1)^2}T_2\ +\ \frac{\mu^2}{n+\nu-1}\ , \end{aligned}$$

and, the Bayes risk of the M.L.E. $\widehat{\theta}$ of $\theta$ is

$$\begin{aligned} r(\mathcal{U}(0,b),\ \widehat{\theta}) &= E^{\theta}[E^{\underline{x}}(\theta - \widehat{\theta} \mid \theta)^2] \\ &= E^{\theta}[E^{\underline{x}}(\theta^2\ -\ 2\theta\widehat{\theta}\ +\ \widehat{\theta}^2 \mid \theta)] \\ &= \frac{b^2}{3n^2}T_3\ . \end{aligned}$$

(ii) Similarly on (i).

The Bayes risk's ratio of the Bayes estimator to the M.L.E. are formed as a function of the mean of the true prior distribution.

## 4. Monte Carlo Simulation Study

In this section, for the type-II progressively censored case, we compare the Bayes estimators with the M.L.E.'s in terms of the M.S.E.'s. We use 300 replications to compute the M.S.E.'s and Bias's of the estimators, the subroutine GGUBS of the packages IMSL to generate uniform random numbers and take a transformation $X = (-\theta \ln U)^{\frac{1}{\gamma}}$ to generate Weibull random numbers.

For the integration, we use the Simpson's composite rule. We obtain the M.L.E.'s of the parameters and reliability function using the Newton-Raphson method.

The M.S.E.'s and Bias's of the estimators are computed for the mission time $t_0(t_o : R(t_0) = 0.1\ (0.2)\ 0.9)$, sample size (N=30,50,80,100), censoring rates(10% , 20% , 30% at each stage), $\theta = 0.5(0.5)1.5$ and $\gamma = 0.35$, when $\gamma$ is known. The M.S.E.'s and Bias's of the estimators are computed for the mission time $t_0(t_o : R(t_0) = 0.1\ (0.2)\ 0.9)$, sample size (N=16,20,28), censoring rates(10% , 20% at each stage), $\theta = (0.5, 1.0) and\ \gamma = (0.35,\ 1.0,\ 1.35)$, when $\gamma$ is unknown.

There are some parts of the simulation results from Table 1 to Table 4. The rest are available based on request. When $\gamma$ is known, the M.L.E.'s for $\theta$ and $R(t_0 \mid \theta, \gamma)$ are denoted by $\hat{\theta}$ and $\widehat{R}$,respectively. The Bayes estimators and G.M.L.E.'s for $\theta$ and $R(t_0 \mid \theta, \gamma)$ under the noninformative prior distribution are denoted by $\theta_1^*, R_1^*$ and $\theta_3^*, R_3^*$, respectively. The Bayes estimators and G.M.L.E.'s for $\theta$ and $R(t_0 \mid \theta, \gamma)$ under the inverted gamma prior distribution are denoted by $\theta_2^*, R_2^*$ and $\theta_4^*, R_4^*$, respectively. When $\gamma$ is unknown, the M.L.E.'s for $\theta, \gamma$ and $R(t_0 \mid \theta, \gamma)$ are denoted by $\tilde{\theta}$, $\tilde{\gamma}$ and $\tilde{R}$, respectively. The Bayes estimators for $\theta$, $\gamma$ and $R(t_0 \mid \theta, \gamma)$ under the inverted gamma prior distribution for $\theta$ and the uniform prior distribution for $\gamma$ are denoted by $\theta_1^{**}$, $\gamma_1^{**}$ and $R_1^{**}$, respectively. The Bayes estimators for $\theta$, $\gamma$ and $R(t_0 \mid \theta, \gamma)$ under both the uniform prior distribution for $\theta$ and $\gamma$ are denoted by $\theta_2^{**}$, $\gamma_2^{**}$ and $R_2^{**}$,respectively. In Figure 1, when the true prior distribution is the uniform distribution, we plot the Bayes risk's ratio of the Bayes estimator to the M.L.E. against the mean of the true prior distribution. In Figure 2, when the true prior distribution is the gamma distribution, we plot the Bayes risk's ratio of the Bayes estimator to the M.L.E. against the mean of the true prior distribution.

(I) Table 1 and Table 2 represent the following facts:

1) For the noninformative prior distribution, the Bayes estimators $\theta_1^*$, $R_1^*$ of $\theta$, $R(t_0 \mid \theta, \gamma)$ have smaller M.S.E. than the G.M.L.E's $\theta_3^*$, $R_3^*$ and M.L.E.'s $\widehat{\theta}$, $\widehat{R}$, respectively.
2) For the inverted gamma prior distribution, the Bayes estimators $\theta_2^*$, $R_2^*$ of $\theta$, $R(t_0 \mid \theta, \gamma)$ have smaller M.S.E. than the G.M.L.E's $\theta_4^*$, $R_4^*$ and M.L.E.'s $\widehat{\theta}$, $\widehat{R}$, respectively.
3) The M.S.E.'s decrease as n increases or a censoring rate decreases, respectively.
4) The estimators of the parameters and reliability function are almost underestimated.

(II) Table 3 and Table 4 represent the following facts:

1) As sample size increases, the Bayes estimators $\theta_1^{**}$, $\theta_2^{**}$ of $\theta$ have smaller M.S.E. than the M.L.E. $\tilde{\theta}$.
2) The Bayes estimators $\gamma_1^{**}$, $\gamma_2^{**}$ of $\gamma$ have smaller M.S.E. than the M.L.E. $\tilde{\gamma}$.
3) The Bayes estimators $R_1^{**}$, $R_2^{**}$ of $R(t_0 \mid \theta, \gamma)$ have smaller M.S.E. than the M.L.E. $\widetilde{R}$.

(III) Figure 1 and Figure 2 represent the following facts.

1) When the true prior distribution is the uniform distribution, the Bayes estimator has smaller Bayes risk than the M.L.E. when the mean of the true prior distribution is more than about 3.

2) When the true prior distribution is the gamma distribution, the Bayes estimator has smaller Bayes risk than the M.L.E. when the mean of the true prior distribution is less than about 1.

3) The Bayes estimator of a scale parameter in $\mathcal{W}(\theta, \gamma)$ using the squared error loss is robust in the sense that the Bayes risk's ratio of the Bayes estimator to the M.L.E. is invariant if the mean of the true prior distribution is invariant.

**Table 1. Bias's and MSE's for Estimators of the Parameter when $\gamma$ is Knowm $(\mu, \nu) = (2, 2)$, $\theta = 0.5$**

(1) 10% censoring case

| | | $\hat{\theta}$ | | $\theta_1^*$ | | $\theta_2^*$ | | $\theta_3^*$ | | $\theta_4^*$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N | n | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| 30 | 24 | -.0677 | .0121 | -.0489 | .0106 | -.0850 | .0142 | -.0050 | .0069 | -.0417 | .0077 |
| 50 | 40 | -.0635 | .0086 | -.0523 | .0076 | -.0741 | .0099 | -.0254 | .0050 | -.0474 | .0062 |
| 80 | 64 | -.0657 | .0075 | -.0588 | .0068 | -.0725 | .0084 | -.0416 | .0048 | -.0553 | .0060 |
| 100 | 80 | -.0655 | .0066 | -.0600 | .0060 | -.0709 | .0073 | -.0462 | .0044 | -.0571 | .0054 |

(2) 30% censoring case

| | | $\hat{\theta}$ | | $\theta_1^*$ | | $\theta_2^*$ | | $\theta_3^*$ | | $\theta_4^*$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N | n | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| 30 | 12 | -.1130 | .0246 | -.0778 | .0202 | -.1428 | .0305 | .0110 | .0102 | -.0571 | .0108 |
| 50 | 20 | -.1067 | .0184 | -.0860 | .0152 | -.1254 | .0221 | -.0302 | .0073 | -.0710 | .0104 |
| 80 | 32 | -.1037 | .0160 | -.0909 | .0139 | -.1157 | .0018 | -.0551 | .0080 | -.0805 | .0109 |
| 100 | 40 | -.1101 | .0161 | -.1001 | .0648 | -.1196 | .0181 | -.0709 | .0088 | -.0908 | .0117 |

**Table 2. Bias's and MSE's for Estimators of the Reliabilty Function when $\gamma$ is Known $(\mu,\nu)=(2,2)$, $\theta=0.5$**

(1) 10% censoring case

| | | | | $\hat{R}$ | | $R(t_1^*)$ | | $R(t_2^*)$ | | $R(t_3^*)$ | | $R(t_4^*)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N | n | t0 | R(t0) | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| | | 1.4950 | .1 | -.0277 | .0020 | -.0179 | .0017 | -.0348 | .0023 | .0412 | .0013 | -.0173 | .0014 |
| | | .2350 | .3 | -.0554 | .0077 | -.0457 | .0066 | -.0690 | .0092 | .0469 | .0045 | -.0341 | .0046 |
| 30 | 24 | .0480 | .5 | -.0601 | .0091 | -.0540 | .0082 | -.0747 | .0111 | .0380 | .0050 | -.0368 | .0050 |
| | | .0070 | .7 | -.0469 | .0056 | -.0444 | .0053 | -.0582 | .0696 | .0240 | .0028 | -.0283 | .0029 |
| | | .0002 | .9 | -.0232 | .0012 | -.0228 | .0011 | -.0127 | .0170 | .0104 | .0003 | -.0490 | .0023 |
| | | 1.4950 | .1 | -.0269 | .0015 | -.0209 | .0013 | -.0314 | .0017 | .0130 | .0012 | -.0203 | .0011 |
| | | .2350 | .3 | -.0507 | .0054 | -.0448 | .0048 | -.0590 | .0063 | .0113 | .0026 | -.0378 | .0038 |
| 50 | 40 | .0480 | .5 | -.0532 | .0060 | -.0496 | .0056 | -.0620 | .0071 | .0068 | .0024 | -.0393 | .0041 |
| | | .0070 | .7 | -.0405 | .0035 | -.0391 | .0034 | -.0472 | .0042 | .0029 | .0012 | -.0297 | .0023 |
| | | .0002 | .9 | -.0162 | .0005 | -.0160 | .0005 | -.0189 | .0006 | .0005 | .0001 | -.0118 | .0003 |
| | | 1.4950 | .1 | -.0283 | .0013 | -.0245 | .0012 | -.0311 | .0015 | -.0041 | .0007 | -.0240 | .0011 |
| | | .2350 | .3 | -.0517 | .0049 | -.0480 | .0043 | -.0570 | .0053 | -.0126 | .0020 | -.0434 | .0037 |
| 80 | 64 | .0480 | .5 | -.0535 | .0052 | -.0513 | .0049 | -.0591 | .0058 | -.0152 | .0021 | -.0446 | .0040 |
| | | .0070 | .7 | -.0403 | .0030 | -.0395 | .0029 | -.0046 | .0034 | -.0124 | .0012 | -.0334 | .0022 |
| | | .0002 | .9 | -.0160 | .0004 | -.0159 | .0004 | -.0177 | .0005 | -.0052 | .0001 | -.0132 | .0003 |
| | | 1.4950 | .1 | -.0285 | .0012 | -.0255 | .0010 | -.0308 | .0013 | -.0092 | .0005 | -.0250 | .0010 |
| | | .2350 | .3 | -.0511 | .0041 | -.0481 | .0038 | -.0553 | .0045 | -.0197 | .0018 | -.0443 | .0033 |
| 100 | 80 | .0480 | .5 | -.0523 | .0044 | -.0505 | .0042 | -.0567 | .0049 | -.0215 | .0018 | -.0452 | .0035 |
| | | .0070 | .7 | -.0391 | .0025 | -.0384 | .0024 | -.0425 | .0028 | -.0167 | .0010 | -.0336 | .0019 |
| | | .0002 | .9 | -.0155 | .0004 | -.0154 | .0004 | -.0168 | .0004 | -.0067 | .0001 | -.0133 | .0003 |

**Table 2. (continued)**

(2) 30% censoring case ($\gamma = 0.35$)

| | | | | $\hat{R}$ | | $R(t_1^*)$ | | $R(t_2^*)$ | | $R(t_3^*)$ | | $R(t_4^*)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| N | n | t0 | R(t0) | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| | | 1.4950 | .1 | -.0435 | .0036 | -.0262 | .0027 | -.0544 | .0042 | .1002 | .0033 | -.0235 | .0019 |
| | | .2350 | .3 | -.0937 | .0164 | -.0740 | .0128 | -.1176 | .0206 | .1129 | .0063 | -.0472 | .0066 |
| 30 | 12 | .0480 | .5 | -.1069 | .0219 | -.0931 | .0184 | -.1351 | .0286 | .0929 | .0072 | -.0509 | .0074 |
| | | .0070 | .7 | -.0867 | .0150 | -.0806 | .0135 | -.1105 | .0202 | .0600 | .0042 | -.0394 | .0044 |
| | | .0002 | .9 | -.0367 | .0028 | -.0358 | .0027 | -.0471 | .0038 | .0213 | .0006 | -.0160 | .0007 |
| | | 1.4950 | .1 | -.0430 | .0028 | -.0322 | .0022 | -.0502 | .0033 | .0372 | .0012 | -.0298 | .0018 |
| | | .2350 | .3 | -.0866 | .0123 | -.0745 | .0102 | -.1018 | .0148 | .0397 | .0053 | -.0570 | .0066 |
| 50 | 20 | .0480 | .5 | -.0950 | .0155 | -.0869 | .0137 | -.1124 | .0191 | .0304 | .0043 | -.0604 | .0076 |
| | | .0070 | .7 | -.0748 | .0100 | -.0714 | .0094 | -.0891 | .0126 | .0183 | .0020 | -.0464 | .0045 |
| | | .0002 | .9 | -.0309 | .0017 | -.0304 | .0017 | -.0369 | .0022 | .0060 | .0003 | -.0187 | .0007 |
| | | 1.4950 | .1 | -.0429 | .0026 | -.0359 | .0021 | -.0475 | .0030 | .0045 | .0012 | -.0341 | .0019 |
| | | .2350 | .3 | -.0835 | .0105 | -.0759 | .0092 | -.0934 | .0121 | -.0038 | .0031 | -.0642 | .0069 |
| 80 | 32 | .0480 | .5 | -.0897 | .0125 | -.0847 | .0115 | -.1008 | .0146 | -.0089 | .0030 | -.0675 | .0078 |
| | | .0070 | .7 | -.0695 | .0077 | -.0674 | .0074 | -.0784 | .0091 | -.0090 | .0017 | -.0515 | .0046 |
| | | .0002 | .9 | -.0282 | .0013 | -.0280 | .0013 | -.0320 | .0015 | -.0042 | .0002 | -.0207 | .0007 |
| | | 1.4950 | .1 | -.0459 | .0027 | -.0404 | .0022 | -.0496 | .0030 | -.0092 | .0009 | -.0386 | .0020 |
| | | .2350 | .3 | -.0883 | .0105 | -.0821 | .0094 | -.0962 | .0119 | -.0240 | .0030 | -.0721 | .0074 |
| 100 | 40 | .0480 | .5 | -.0941 | .0123 | -.0900 | .0114 | -.1030 | .0140 | -.0279 | .0032 | -.0755 | .0083 |
| | | .0070 | .7 | -.0724 | .0074 | -.0707 | .0071 | -.0796 | .0086 | -.0224 | .0019 | -.0573 | .0049 |
| | | .0002 | .9 | -.0293 | .0012 | -.0291 | .0012 | -.0323 | .0014 | -.0093 | .0003 | -.0230 | .0008 |

**Table 3. Bias's and MSE's for Estimators of the Parameters when $\gamma$ is Unknown $(\mu,\nu)=(0.5,0.5)$, (a,b)=(0,1), $\delta=4$, $\theta=0.5$**

(1) 10% censoring case ($\gamma = 0.35$)

| | | $\tilde{\theta}/\tilde{\gamma}$ | | $\theta_1^{**}/\gamma_1^{**}$ | | $\theta_2^{**}/\gamma_2^{**}$ | |
|---|---|---|---|---|---|---|---|
| N | n | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| 16 | 12 | -.1520 | .0418 | -.0787 | .0215 | .1758 | .0310 |
| | | .1151 | .0285 | .1056 | .0196 | .0112 | .0057 |
| 20 | 16 | -.1324 | .0305 | -.0848 | .0183 | .1689 | .0285 |
| | | .0876 | .0197 | .0912 | .0162 | .0030 | .0048 |
| 28 | 22 | -.1258 | .0273 | -.0838 | .0171 | .0038 | .0001 |
| | | .0719 | .0113 | .0667 | .0090 | .0259 | .0042 |

(2) 10% censoring case ($\gamma = 1.0$)

| | | $\tilde{\theta}/\tilde{\gamma}$ | | $\theta_1^{**}/\gamma_1^{**}$ | | $\theta_2^{**}/\gamma_2^{**}$ | |
|---|---|---|---|---|---|---|---|
| N | n | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| 16 | 12 | -.1520 | .0418 | .0665 | .0172 | .1845 | .0344 |
| | | .3290 | .2330 | -.1416 | .0217 | -.1841 | .0370 |
| 20 | 16 | -.1324 | .0305 | .0311 | .0097 | .1709 | .0292 |
| | | .2504 | .1612 | -.1260 | .0177 | -.1724 | .0332 |
| 28 | 22 | -.1252 | .0273 | .0095 | .0085 | .0128 | .0008 |
| | | .2055 | .0922 | -.1058 | .0123 | -.1238 | .0170 |

**Table 4. Bias's and MSE's for Estimators of the Reliability Function when $\gamma$ is Unknown $(\mu,\nu)=(0.5,0.5)$, (a,b)=(0,1), $\delta=4$, $\theta=0.5$**

(1) 10% censoring case ($\gamma=0.35$)

| | | | | $\tilde{R}$ | | R1** | | R2** | |
|---|---|---|---|---|---|---|---|---|---|
| N | n | t0 | R(t0) | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| | | 1.4950 | .1 | -.0555 | .0051 | -.0282 | .0029 | .0799 | .0044 |
| | | 0.2350 | .3 | -.0845 | .0180 | -.0343 | .0094 | .1137 | .0144 |
| 16 | 12 | 0.0480 | .5 | -.0431 | .0182 | .0031 | .0115 | .0991 | .0140 |
| | | 0.0070 | .7 | .0088 | .0119 | .0295 | .0094 | .0577 | .0077 |
| | | 0.0002 | .9 | .0207 | .0033 | .0184 | .0029 | .0072 | .0019 |
| | | 1.4950 | .1 | -.0516 | .0041 | -.0322 | .0025 | .0784 | .0040 |
| | | 0.2350 | .3 | -.0745 | .0133 | -.0391 | .0079 | .1073 | .0128 |
| 20 | 16 | 0.0480 | .5 | -.0388 | .0131 | -.0043 | .0094 | .0915 | .0120 |
| | | 0.0070 | .7 | .0023 | .0087 | .0217 | .0082 | .0521 | .0068 |
| | | .0002 | .9 | .0133 | .0030 | .0153 | .0026 | .0056 | .0018 |
| | | 1.4950 | .1 | -.0500 | .0039 | -.0323 | .0024 | -.0005 | .0000 |
| | | 0.2350 | .3 | -.0735 | .0126 | -.0463 | .0081 | .0158 | .0012 |
| 28 | 22 | 0.0480 | .5 | -.0439 | .0114 | -.0216 | .0084 | .0246 | .0040 |
| | | 0.0070 | .7 | -.0040 | .0069 | .0044 | .0058 | .0189 | .0043 |
| | | 0.0002 | .9 | .0125 | .0019 | .0097 | .0016 | .0027 | .0016 |

(2) 10% censoring case ($\gamma=1.0$)

| | | | | $\tilde{R}$ | | R1** | | R2** | |
|---|---|---|---|---|---|---|---|---|---|
| N | n | t0 | R(t0) | BIAS | MSE | BIAS | MSE | BIAS | MSE |
| | | 1.151 | .1 | -.0555 | .0051 | .0363 | .0036 | .0921 | .0050 |
| | | 0.602 | .3 | -.0845 | .0180 | -.0004 | .0048 | .0770 | .0061 |
| 16 | 12 | 0.347 | .5 | -.0435 | .0182 | -.0377 | .0058 | .0357 | .0060 |
| | | 0.178 | .7 | .0083 | .0120 | -.0605 | .0076 | -.0076 | .0008 |
| | | 0.053 | .9 | .0207 | .0034 | -.0512 | .0038 | -.0326 | .0016 |
| | | 1.151 | .1 | -.0516 | .0041 | .0205 | .0020 | .0869 | .0041 |
| | | 0.602 | .3 | -.0745 | .0133 | -.0149 | .0041 | .0747 | .0057 |
| 20 | 16 | 0.347 | .5 | -.0392 | .0131 | -.0473 | .0058 | .0361 | .0059 |
| | | 0.178 | .7 | .0019 | .0098 | -.0641 | .0076 | -.0048 | .0007 |
| | | 0.053 | .9 | .0133 | .0030 | -.0493 | .0035 | -.0289 | .0014 |
| | | 1.151 | .1 | -.0501 | .0039 | .0102 | .0017 | .0096 | .0002 |
| | | 0.602 | .3 | -.0735 | .0126 | -.0233 | .0045 | -.0161 | .0005 |
| 28 | 22 | 0.347 | .5 | -.0442 | .0114 | -.0506 | .0070 | -.0409 | .0021 |
| | | 0.178 | .7 | -.0045 | .0070 | -.0618 | .0070 | -.0546 | .0035 |
| | | 0.053 | .9 | .0125 | .0022 | -.0438 | .0027 | -.0429 | .0022 |

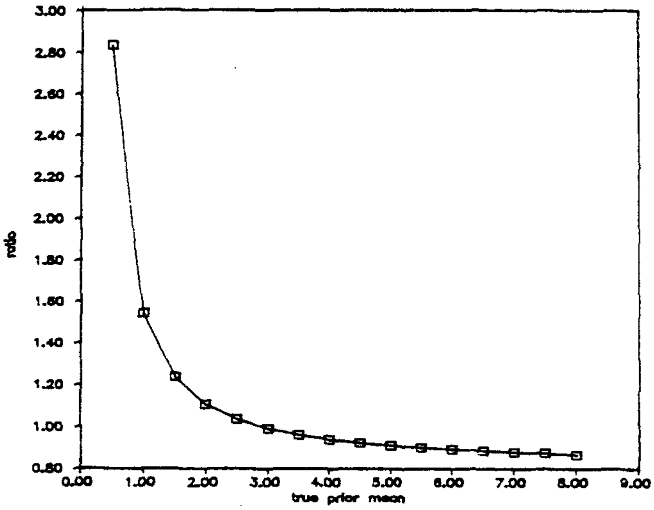


**Figure 1. Bayes Risk's Ratio of the Bayes Estimator to the MLE when the True Prior is the Uniform Distribution**

$(\mu, \nu) = (3, 2),\ n = 80,\ \sum_{i=1}^{n} \gamma_i = 20$

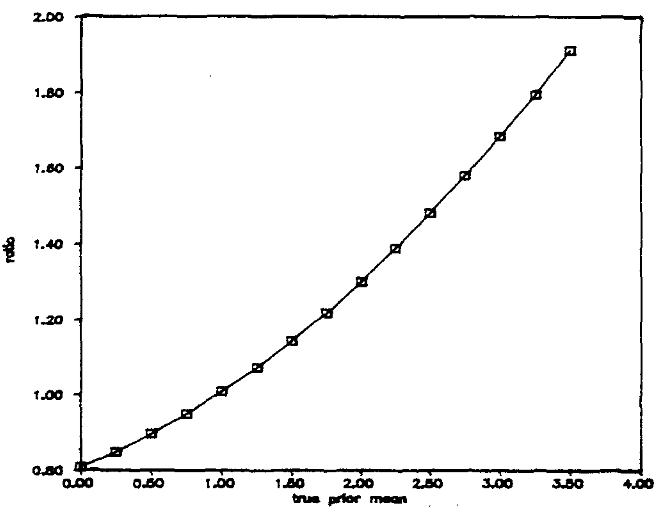


**Figure 2. Bayes Risk's Ratio of the Bayes Estimator to the MLE when the True Prior is the Gamma Distribution**

$(\mu, \nu) = (3, 2),\ n = 80,\ \sum_{i=1}^{n} \gamma_i = 20,\ \alpha = 1$

## References


1. Basu, A.P.(1964). Estimation of Reliability for some Distribution useful in Life Testing, *Technometrics*, 6, 215-219.
2. Bury, K.V.(1972). Bayesian Decision Analysis of the Hazard Rate for a Two-parameter Weibull Process, *IEEE Transaction on Reliability*, R-21, 159-169.
3. Box, G.P. and Tiao. G.C.(1973). *Bayesian Inference in Statistical Analysis*, Addison Wesley, New York.
4. Caciari, M. and Montanari, G.C.(1987). A Method to Estimate the Weibull Parameters for Progressively Censored Tests, *IEEE Transaction on Reliability R-36*, 87-93.
5. Cohen, A.C.(1963). Progressively Censored Samples in Life Testing, *Technometrics*, 5, 327-339.
6. Cohen, A.C.(1965). Maximum Likelihood Estimation in Weibull Distribution Based on Complete and on Censored Samples, *Technometrics*, 7, 579-588.
7. Gibbons, D.I. and Vance, L.C.(1983). Estimators for the 2-parameter Weibull Distribution with Progressively Censored Samples, *IEEE Transaction on Reliability R-32*, 95-99.
8. John, L.M. and James, C.S.(1986). On the Sensitivity of Posterior to Prior in Bayesian Analysis.*American Statistical Association: Proceedings of the Business and Economic statistics section*, 131-134.
9. Martz, H.F. and Waller, R.A.(1982). *Bayesian Reliability Analysis,* Wiley, New York.
10. Miller, R.G.(1981). *Survival Analysis*, Wiley, New York.
11. Wang, M.C.(1987). Product Limit estimate : A Generalized Maximum Likelihood Study, *Comunications in Statistics, Theory and Methods. 16*, 3117-3132.