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On the Minimal Generating Systems of Modules and Projective Modules over Semilocal Domains

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§1. Introduction

In the late 1950's, the emergence of algebraic geometry has contributed to the development of theories of rings in the study of algebra ([5], [10]). For instance, the theories of ideal have been studied in terms of algebriac geometry ([4], [8], [11], [12]), and the projective modules on polynomial rings, too ([9], [14], [16]). The condition under which a projective module becomes a free module has been obtained with the help of algebraic geometry, and it was formulated by Quillen and Suslin in 1976 as follows ([13], [15]).

Theorem 1.1. (Quillen-Suslin). Let R be a commutative ring with 1. If M is a finitely generated projective $R[X]$-module, $f\in R[X]$ a monic polynomial such that M_f is a free $R[X]_f$-module, then M is a free $R[X]$-Module.

In section 2, the terms and notations are briefly illustrated, since these will be used in section 3 and section 4.

And a property of local rings is proved in Theorem 2.8.

In sections 3, a property of the minimal generating system of module is proved in Theorem 3.7.

In section 4, using Quillen-Suslin's Theorem 1.1, we prove Theorem 4.7, which states the following:

Let R be a semilocal domain with maximal ideals $\mathfrak{m}_1, \mathfrak{m}, \cdots, \mathfrak{m}_t$ such that each local ring $R_{\mathfrak{m}_i}$ is a principal ideal domain for $i=1,2,\cdots,t$. Then all finitely generated projective $R[X_1,\cdots,X_n]$-modules are free.

§ 2. Preliminaries

In this paper, by a ring we mean a commutative ring with 1. A ring R is called Noetherian if any ideal of R has a finite system of generators. Every principal ring, in particular, every field is a Noetherian ring. Moreover, if R is a Noetherian ring, then the polynomial ring $R]X_1, X_2, \cdots, X_n]$ is also Noetherian ([10]).

Lemma 2.1. If R is a principal ideal domain, then every submodule $U \subset R^n$ is a free R-module with a finite rank $(\leq n)$.

Proof. When $n=1$, a submodule $U \subset R$ is an ideal of R. By our hypothesis, $U=uR$ $(u \in U)$. We assume that $n>1$, and that our assertion holds for $n-1$.

Consider the elements $u=(u_1, \cdots, u_n) \in U \subset R^n$.

(i) If $u_1=0$ for every element $u \in U$, then $U \subset R^{n-1}$ and thus U is a free R-module by our induction on hypothesis.

(ii) Let $u_1 \neq 0$. Let $\mathfrak{a}$ be the ideal of R consisting of all elements u_1 which are the first components of all element in U. By our hypothesis, we have an element $u \in R$ such that $\mathfrak{a}=uR$. We put

$$\bar{\mathfrak{a}}=\{(u_1, 0, 0, \cdots, 0) \in R^n \mid u_1 \in \mathfrak{a}\}.$$

Then it is a submodule of R^n. It is clear that

$$U-\bar{\mathfrak{a}} \subset R^{n-1}$$

which is a free R-module with a basis $\{u_2, \cdots, u_n\}$. It follows from the above description that $\{u, u_2, \cdots, u_n\}$ is a basis of U. ///

For a ring R, let S be a multiplicative closed subset of R.

For a R-module M and the cannonical mapping $i: M \longrightarrow M_S$,

$$\text{Ker } (i)=\{m \in M \mid \text{there is an } s \in S \text{ such that } sm=0\}$$

Therefore i is injective if and only if there is no element s in S such that $sm=0$ for some $m(\neq 0) \in M$. Accordingly, $i: R \longrightarrow R_S$ is injective if any only if S contains no zero divisor of R.

Proposition 2.2. With the above notations,

(i) $M_S=\{0\}$ if and only if for any $m \in M$, there is an $s \in S$ with $sm=0$, and $R_S=\{0\}$

if only if $0 \in S$,

(ii) For $f \in R$, $R_f = \{0\}$ if any only f is nilpotent,

(iii) $M = \{0\}$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R.

Proof. (i) Since $\frac{m}{s} \in M_S$ is zero if there is an element $s' \in S$ such that $s'm = 0$, $M_S = \{0\}$ if and only if there is an $s \in S$ such that $sm = 0$ for each $m \in M$. Thus $R_S = \{0\}$ if and only if $0 \in S$ because $1 \in R$.

(ii) Each element of R_f is of the form $\frac{r}{f^n}$ where $r \in R$. Since $\frac{1}{f} = 0$ if and only if there exists a positive integer m such that $f^m = 0$, f is a nilpotent element.

(iii) Suppose $M_{\mathfrak{m}} = \{0\}$ for all maximal ideals $\mathfrak{m}$ of R. Then, by (i), for any $m \in M$ Ann(m) (the set of all anihilators in R pf m) is contained in no maximal ideals of R. That is, Ann(m) contains a unit of R. Therefore, Ann(m) $= R \ni 1$. Hence $m = 1 \cdot m = 0$. ///

Let S be a multiplicative closed subset of a ring R, and let M be a R-module. It is clear that a submodule U of M is always contained in the kernel of the composite mapping

$$M \xrightarrow{\alpha} M_S \xrightarrow{\beta} M_S/U_S$$

where α and β are the cannonical mappings

Proposition 2.3. $\rho : (M/U)_S \longrightarrow M_S/U_S$ defined by

$$\rho\left(\frac{m+U}{s}\right) = \frac{m}{s} + U_S \quad (m \in M, \ s \in S)$$

is an isomorphism.

Proof. It is clear that ρ is surjective. Thus we have to show that Ker $\rho = \{0\}$. Assume that

$$\rho\left(\frac{m+U}{s}\right) = 0 \quad (m \in M, \ s \in S).$$

Then, by the above definition, $\frac{m}{s} \in U_S$. This means that there exist $u \in U$ and $s' \in S$ such $\frac{m}{s} = \frac{u}{s'}$. In consequence, there exists $s'' \in S$ such that $s''(s'm - su) = 0$.

Hence we have the following :

$$\frac{m+U}{s} = \frac{s''s'm+U}{s''ss'} = \frac{s''su+U}{s''ss'} = 0 \text{ in } (M/U)_S.$$

Lemma 2.4. With the above notations for submodules P and Q of M, $P=Q$ if and only if for all maximal ideals $\mathfrak{m}$ of R $P_{\mathfrak{m}}=Q_{\mathfrak{m}}$.

Proof. By Proposition 2.3, for each maximal ideal $\mathfrak{m}$ of R, we have the following:

$$\left(\frac{P+Q}{Q}\right)_{\mathfrak{m}}=\frac{P_{\mathfrak{m}}+Q_{\mathfrak{m}}}{Q_{\mathfrak{m}}} \text{ and } \left(\frac{P+Q}{P}\right)_{\mathfrak{m}}=\frac{P_{\mathfrak{m}}+Q_{\mathfrak{m}}}{P_{\mathfrak{m}}}.$$

If $P_{\mathfrak{m}}=Q_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R, then by (iii) of Proposition 2.2,

$$\frac{P+Q}{Q}=\frac{P+Q}{P}=\{0\}.$$

That is, $Q=P+Q=P$.
It is obvious that $P=Q$ implies $P_{\mathfrak{m}}=Q_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R. ///

Lemma 2.5. (Nakayama's Lemma). Let an ideal $\mathfrak{a}$ of R be contained in $\bigcap_{\mathfrak{m}:\text{maximal in } R}\mathfrak{m}$. For a R-module M and a submodule N of M such that M/N is finitely generated, if $M=N+\mathfrak{a}M$, then $M=N$. ///

Definition 2.6. For a ring R and a R-module M, we define the following :

(i) Spec $(R)=\{\mathfrak{p}\subsetneq R \mid \mathfrak{p}$ is a prime ideal of $R\}$ with the Zariski topology (or the Spectrum topology).

(ii) $J(R)=\{\mathfrak{p}\in$ Spec $(R)\mid\mathfrak{p}$ can be written as the intersection of maximal ideals$\}$ with the relative topology, which is called the *J-spectrum* of R.

(iii) Max$(R)=\{\mathfrak{m}\in$ Spec $(R)\mid\mathfrak{m}$ is a maximal ideal$\}$ with the relative topology, which is called the *maximal spectrum* of R.

(iv) Supp$(M)=\{\mathfrak{p}\in$ Spec $(R)\mid M_{\mathfrak{p}}\neq\{0\}\}$.

Obviously, Max $(R)\subset J(R)\subset$ Spec (R). If X is one of these space and $\mathfrak{a}$ is an ideal of R.

$$\mathfrak{V}(\mathfrak{a})=\{\mathfrak{p}\in X\mid\mathfrak{p}\supset\mathfrak{a}\}$$

is called the *zero set* of $\mathfrak{a}$ in X.

Proposition2.7. If M is finitely generated, then

$$\text{Supp }(M)=\mathfrak{V}(\text{Ann } M)).$$

In particular, Supp (M) is a closed subset of Spec (R).

Proof. Let $\{m_1, \cdots, m_t\} \subset M$ be a set of generators. Then $\mathfrak{p} \notin \mathrm{Supp}\ (M)$ implies $M_{\mathfrak{p}} = \{0\}$. By (i) of Proposition 2. 2, there exist elements $s_i \in R - \mathfrak{p}$ such that

$$s_i m_i = 0 \ (i = 1, 2, \cdots, t).$$

We put $s = s_1 s_2 \cdots s_t$. Then $s \in \mathrm{Ann}\ (M)$ and $s \notin \mathfrak{p}$.
Thus $\mathfrak{p} \notin \mathfrak{V}(\mathrm{Ann}\ (M))$ by our definition above.

Conversely, we suppose that $\mathfrak{p} \notin \mathfrak{V}(\mathrm{Ann}\ (M))$. This means that there exists an element $s \notin \mathrm{Ann}\ (M)$ with $s \notin \mathfrak{p}$. By (i) of Proposition 2. 2, $M_{\mathfrak{p}} = \{0\}$. That is, $\mathfrak{p} \notin \mathrm{Supp}\ (M)$. ///

Let X be a topological space. X is said to be *irreducible* if for any decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subset X$ $(i = 1, 2)$ we have $X = A_1$ or $X = A_2$. X is said to be Noetherian if every descending chain $A_1 \supset A_2 \supset \cdots$ of closed subsets A_i of X is stationary. An *irreducible component* of X is a maximal irreducible subset of X. It is well-known that every Noetherian topological space has only finitely many irreducible components.

The *Krull dimension* $\dim R$ of a ring R is the dimension of Spec (R) i. e., the supremum of the lengths n of all chains

$$X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n \qquad (*)$$

of nonempty closed irreducible subsets X_i of X if we put $X = \mathrm{Spec}\ (R)$. This is just the supremum of the lengths n of all prime ideal chains

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \qquad (**)$$

in Spec (R). The height ht $(\mathfrak{p})$ of $\mathfrak{p} \in \mathrm{Spec}\ (R)$ is the supremum of the lengths of all chains $(**)$ with $\mathfrak{p} = \mathfrak{p}_n$.

For an arbitrary ideal $\mathfrak{a} \neq R$, the dimension of $\mathfrak{a}$, written $\dim \mathfrak{a}$, is just $\dim(R/\mathfrak{a}) = \dim\ (\mathrm{Spec}\ (R/\mathfrak{a}))$. Moreover, for each $\mathfrak{p} \in \mathrm{Spec}\ (R)$, we can prove that

$$\dim\ (\mathfrak{p}) = \dim\ (\mathfrak{V}(\mathfrak{p}))$$

([10]).

For a ring homomorphism $\alpha : R \longrightarrow S$, it is well-hnown that the mapping

$$\begin{array}{cccc} \mathrm{Spec}\ (\alpha) = \varphi : & \mathrm{Spec}\ (S) & \longrightarrow & \mathrm{Spec}\ (R) \\ & \mathbin{\rotatebox[origin=c]{90}{$\in$}} & & \mathbin{\rotatebox[origin=c]{90}{$\in$}} \\ & q & \longmapsto & \alpha^{-1}(q), \end{array}$$

is continuous. We put

$$S_{\mathfrak{p}} = \text{the ring of fractions of } S \text{ with denominator set } \alpha(R \setminus \mathfrak{p})$$

where $\mathfrak{p} \in \text{Spec }(R)$.

Theorem 2.8. Under the above situation, for each $\mathfrak{p} \in \text{Spec }(R)$ we have the followings.

(i) The elements of $\varphi^{-1}(\mathfrak{p})$ correspond bijectively with the elements of the fibre Spec $(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}})$ of φ over $\mathfrak{p}$.

(ii) If S is a finitely generated R-module, then the number of elements of $\varphi^{-1}(\mathfrak{p})$ is at most as large as the deimension of $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ as a vector space $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ (a field).

Proof. (i) We have to note that

$$\varphi^{-1}(\mathfrak{p}) = \{q \in \text{Spec }(S) \mid q \cap \alpha(R) = \alpha(\mathfrak{p})\}$$

and Spec $(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) = \{q_{\mathfrak{p}} \mid q \in \varphi^{-1}(\mathfrak{p})\}$ where $q_{\mathfrak{p}}$ is the ideal of fractions of $q \in \varphi^{-1}(\mathfrak{p})$ with denominator set $\alpha(R \setminus \mathfrak{p})$. Therefore we have an one-to-one and onto correspondence between $\varphi^{-1}(\mathfrak{p})$ and Spec $(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}})$.

(ii) Since S is finitely generated as R-module, S is integral over R. For any two different elements q_1 and q_2 in $\varphi^{-1}(\mathfrak{p})$, $q_1 \cap q_2 = \alpha(\mathfrak{p})$ ([3], [5], [10]).

Moreover, $S_{\mathfrak{p}}$ is also finitely generated as a $R_{\mathfrak{p}}$-module.

We assum that

$$S_{\mathfrak{p}} = R_{\mathfrak{p}}s_1 + \cdots + R_{\mathfrak{p}}s_t \quad (s_i \in S_{\mathfrak{p}} \text{ for } i = 1, 2, \cdots, t),$$

and that

$$s_i \notin \mathfrak{p}S_{\mathfrak{p}} \text{ for } i = 1, 2, \cdots, r$$
$$s_j \in \mathfrak{p}S_{\mathfrak{p}} \text{ for } j = r+1, \cdots, t.$$

Then the dimension of $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ over the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is just r. Moreover, for the cannonical map $\psi : S \longrightarrow S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ as in the diagram

$$\begin{array}{ccc} S & \longrightarrow & S_{\mathfrak{p}} \\ & \psi \searrow \quad © \quad \swarrow & \\ & S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} & \end{array}$$

Spec (ψ) is one-to-one and into. Since $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a field. every ideal of $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is generated by a subset of $\{s_1, \cdots, s_r\}$. Therefore, the number of ideals such that any two ideals meet only on $\{0\}$ is r or less than r. Thereeore, the number of prime ideals in $\varphi^{-1}(\mathfrak{p}) \leq r$. ///

§3. Minimal Generating Systems

Let $R(\neq 0)$ be a ring and let M be a finitely generated R-module. We put

$\mu(M)$ = the number of elements in a shortest system of generators of M, which is called a *minimal generating system* of M over R.

It is clear that $\mu(M)$ is a fixed number for M, and also that if M is a finitely generated free R-module, then the minimal generating systems are just the bases of M ([10]).

Lemma 3.1. For a local ring $(R, \mathfrak{m})$, let M be a finitely generated R-module.

(i) $\mu(M) = \dim_K(M/\mathfrak{m}M)$ where $K = R/\mathfrak{m}$ is a field.

(ii) If $\{m_1, \cdots, m_t\}$ is a minimal generating system of M and if there exist $r_1, \cdots, r_t \in R$ such that

$$\sum_{i=1}^{t} r_i m_i = 0,$$

then $r_i \in \mathfrak{m}$ for $i = 1, 2, \cdots, t$.

(iii) Any generating system of M contains a minimal generating system.

Proof. (i) For $\{m_1, \cdots, m_t\} \subset M$, we assume that

$$M \supset Rm_1 + Rm_2 + \cdots + Rm_t.$$

Suppose the cannonical mapping

$$M \longrightarrow M/\mathfrak{m}M \quad (m_i \longmapsto \bar{m}_i \ \forall i = 1, 2, \cdots, t),$$

and $M/\mathfrak{m}M = K\bar{m}_i + \cdots + K\bar{m}_t$.

Then, it follows that $M = Rm_1 + \cdots + Rm_t + \mathfrak{m}M$.

By Lemma 2.5 (Nakayama's Lemma), $\mathfrak{m}M = \{0\}$. Therefore, $\{m_1, \cdots, m_t\}$ is a minimal generating system of M if and only if $\{\bar{m}_1, \cdots, \bar{m}_t\}$ is a basis of $M/\mathfrak{m}M$ over the field K, and thus $\mu(M)=\dim_K(M/\mathfrak{m}M)$.

(ii) Let $\{m_1,\cdots,m_t\}$ be a minimal generating system of M. Then, as in the proof of (i), $\{\bar{m}_1,\cdots,\bar{m}_t\}$ is a basis of $M/\mathfrak{m}M$ over K.

$$\text{If } \sum_{i=1}^{t} r_i m_i=0, \text{ then } \sum_{i=1}^{t} \bar{r}_i\bar{m}_i=0.$$

This implies $\bar{r}_i=0$ in K for $i=1,2,\cdots,t$.
Since $K=R/\mathfrak{m}$, $\bar{r}_i=0$ implies that $r_i\in\mathfrak{m}$ for all $i=1,2,\cdots,t$.

(iii) We assume that

$$M=Rm_1+\cdots+Rm_t \quad (m_i\in M \text{ for } i=1,2,\cdots,t).$$

Then, as before, $\{\bar{m}_1,\cdots,\bar{m}_t\}$ is a generating system of $M/\mathfrak{m}M$.

Thus we have a basis $\{\bar{m}_{j_1},\cdots,\bar{m}_{j_s}\}\subset\{\bar{m}_1,\cdots,\bar{m}_t\}$ of $M/\mathfrak{m}M$ over K. Then $\{m_{j_1},\cdots,m_{j_s}\}$ is a minimal generating ststem of M. ///

For a finitely generated R-module M, we put

$$\mu_{\mathfrak{p}}(M)=\text{the number of elements in a shortest generating system of the } R_{\mathfrak{p}}\text{-module } M_{\mathfrak{p}}.$$

for each $\mathfrak{p}\in\operatorname{Spec}(R)$. Further, for $r\in\mathbf{N}(=\{0,1,2,3,4,\cdots\})$, we define an ideal

$$I(M,r)=\sum_{\{m_1,\cdots,m_r\}\subset M}\operatorname{Ann}(M/<m_1,\cdots,m_r>),$$

where the sum is taken over all subsets of M, consisting of r elements. It is clear that (a) $I(M,0)=\operatorname{Ann}(M)$, (b) $I(M,r)\subset I(M,r+1)$ for all $r\in\mathbf{N}$, (c) $I(M,r)=R$ if $r\geq\mu(M)$ and (d) $I(M_S,r)=I(M,r)_S$ for any multiplicative closed set $S\subset R$.

Proposition 3.2. For $\mathfrak{p}\in\operatorname{Spec}(R)$,

$$\mu_{\mathfrak{p}}(M)\geq r+1 \text{ if and only if } \mathfrak{p}\subset I(M,r).$$

Proof. Suppose $\mu_{\mathfrak{p}}(M)\geq r+1$. If $I(M,r)\not\subset\mathfrak{p}$, then $I(M,r)_{\mathfrak{p}}=R_{\mathfrak{p}}$ and thus $I(M_{\mathfrak{p}},r)=R_{\mathfrak{p}}$. Then, by (c), $\mu(M_{\mathfrak{p}})\leq r$. This contradicts to our asumption $\mu_{\mathfrak{p}}(M)=\mu(M_{\mathfrak{p}})\geq r+1$.

Therefore $I(M,r)\subset\mathfrak{p}$.

Conversely, suppose $\mathfrak{p}\supset I(M,r)$. If $\mu_{\mathfrak{p}}(M)\leq r$, then $I(M_{\mathfrak{p}},r)=R_{\mathfrak{p}}$. Hence $I(M,r)_{\mathfrak{p}}$

$=R_{\mathfrak{p}}$ and $I(M,r)\not\subset\mathfrak{p}$. This is a contradiction. ///

Definition 3.3. (i) Let M be a finitely generated R-module. An element $m\in M$ is said to be *basic* at $\mathfrak{p}\in\mathrm{Spec}\,(R)$ if $\bar{m}\notin\mathfrak{p}M_{\mathfrak{p}}$, where $M\longrightarrow M_{\mathfrak{p}}(m\longrightarrow\bar{m})$.

(ii) A submodule $U\subset M$ is called *k-times basic* for some $k\in\mathrm{N}$ at $\mathfrak{p}\in\mathrm{Spec}\,(R)$ if $\mu_{\mathfrak{p}}(M)-\mu_{\mathfrak{p}}(M/U)\geq k$.

Let X be the J-spectrum of R and let M be a finitely generated R-module. For each $m\in M$, we define that

$$X(m)=\{\mathfrak{p}\in X\,|\,m \text{ is basic at } \mathfrak{p}\}$$

By Lemma 3.1, $m(\in M)$ is basic at $\mathfrak{p}\in\mathfrak{B}(\mathfrak{a})$ if and only if the image $\bar{m}$ of m in the R-module $M/\mathfrak{a}M$ is basic at $\mathfrak{p}$, where $\mathfrak{a}$ is an ideal of R.

By the above definitions, we have the following properties ([10]).

Property 3.4. With the above notations, the followings hold.

(i) If X is Noetherian and $d=\dim X<\infty$, then $X(m)\cap\mathfrak{B}(\mathfrak{a})$ has only finitely many minimal elements.

(ii) Under the hypothesis of (i), we put

$$u_m=\mathrm{Max}\ \{\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X(m)\}.$$

Then there are only finitely many $\mathfrak{p}\in X(m)$ with

$$\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})=u_m.$$

(iii) Under the hypothesis of (i), let $M=Rm_1+Rm_2+\cdots+Rm_t=\langle m_1,m_2,\cdots,m_t\rangle$ ($m_i\in M$ for $i=1,2,\cdots,t$) and

$$\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})<t \text{ for all } \mathfrak{p}\in X(m_t).$$

Then there exist elements $a_1,a_2,\cdots,a_{t-1}\in R$ such that

$$M=\langle m_1+a_1m_t,\ m_2+a_2m_t,\cdots,m_{t-1}+a_{t-1}m_t\rangle.$$

Proposition 3.5. Let $X=J(R)$ be Noetherian of finite Krull dimension. Then, for a finitely generated R-module M,

$$\mu(M)\leq u=\max\ \{\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X\cap\ \mathrm{Supp}\ (M)\}.$$

Proof. Let $\mu(M)=t$ and $M=<m_1, m_2, \cdots, m_t>$.
We put

$$u=\text{Max}\ \{\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X\cap\ \text{Supp}\ (M)\}$$
$$(=\text{Max}\ \{\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X(m_t)\}).$$

If $u\geq t$, then we have nothing to prove.
Assume that $u<t$. Then

$$\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})<t \text{ for all } \mathfrak{p}\in X(m_t).$$

By (iii) of property 3.4, there are $a_1, \cdots, a_{t-1}\in R$ such that

$$M=<m_1+a_1m_t,\ \ m_2+a_2m_t, \cdots, m_{t-1}+a_{t-1}m_t>.$$

Hence $\mu(M)\leq t-1$. This is a contradiction. ///

Corollary 3.6. (i) Let $\mathfrak{a}$ be an ideal of R and

$$u=Max\ \{\mu_{\mathfrak{p}}(\mathfrak{a})+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in\mathfrak{B}(\mathfrak{a})\}.$$

Then $\mu(\mathfrak{a})\leq\text{Max}\ \{u,\ d+1\}$, where $\dim\ J(R)=d$.

(ii) Let R be a semilocal ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ and let M be a finitely generated R-module, $u=\text{Max}\ \{\mu_{\mathfrak{m}_i}(M)\,|\,i=1, 2, \cdots, t\}$. Then $\mu(M)\leq u$.

Proof. (i) For each $\mathfrak{p}\in\text{Spec}\ (R)$ such that $\mathfrak{a}\not\subset\mathfrak{p}$, we have $\mathfrak{a}_{\mathfrak{p}}=R_{\mathfrak{p}}$. Thus, in this case $\mu_{\mathfrak{p}}(\mathfrak{a})=1$. Moreover, $\dim\ \mathfrak{B}(\mathfrak{p})\leq\dim\ J(R)$. Therefore, we have

$$\text{Max}\ \{\mu_{\mathfrak{p}}(\mathfrak{a})+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{a}\not\subset\mathfrak{p}\in J(R)\}\leq d+1$$

Therefore,

$$\text{Max}\ \{\mu_{\mathfrak{p}}(\mathfrak{a})+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X\cap\text{Supp}\ (\mathfrak{a})\}\leq\text{Max}\ \{u, d+1\}.$$

By proposition 3.5, $\mu(\mathfrak{a})\leq\text{Max}\ \{u,\ d+1\}$.

(ii) Since R is a semilocal ring, $X=J(R)=\text{Max}\ (R)$and $\dim\ J(R)=\dim\ \text{Max}\ (R)=0$ ([6],[10]). Thus, in the formula

$$u=\text{Max}\ \{\mu_{\mathfrak{p}}(M)+\dim\ \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in X\cap\ \text{Supp}\ (M)\}$$

in Proposition 3.5, we have dim $\mathfrak{B}(\mathfrak{p})=0$ for all $\mathfrak{p}\in X\cap$ Supp(M). Moreover, since $\mathfrak{p}\in X\cap$ Supp (M) implies that $\mathfrak{p}$ is maximal ideal, $\mu(M)\leq \operatorname{Max}_{i=1,2,\cdots,t}\{\mu_{\mathfrak{m}_i}(M)\}$. ///

Theorem 3.7. Ket $(R,\mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}\subset\mathfrak{m}$ be an ideal of R with $\xi\in\mathfrak{m}/\mathfrak{a}$ which is not a zero divisor in $R/\mathfrak{a}$. We put $\xi=x+\mathfrak{a}$ and $\bar{R}=R/(x)$ and consider the cannonical mapping

$$\eta : R\longrightarrow R/(x)\ \ (a\longmapsto\bar{a}).$$

Then, (i) $\mathfrak{a}=<a_1,a_2,\cdots,a_t>$ if and only if $\bar{\mathfrak{a}}=<\bar{a}_1,\bar{a}_2,\cdots,\bar{a}_t>$.

(ii) $\mu(\mathfrak{a})=\mu(\bar{\mathfrak{a}})$ where $\eta(\mathfrak{a})=\bar{\mathfrak{a}}$.

Proof. Since $\xi\in\mathfrak{m}/\mathfrak{a}$ and $\xi=x+\mathfrak{a}$, $x\in\mathfrak{m}$. By our hypothesis, there is not any element y in $R\setminus\mathfrak{a}$ such that $yx\in\mathfrak{a}$. Since $(x)\subset\mathfrak{m}$, $\bar{R}$ is a local ring with it maximal ideal $\mathfrak{m}/(x)=\bar{\mathfrak{m}}$.

It follows that $\bar{\mathfrak{a}}\subset\bar{\mathfrak{m}}$.

Since R is a Noetherian local ring, so is $\bar{R}$. Hence $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ are finitely generated ideals. We assume $\mathfrak{a}=<m_1,m_2,\cdots,m_r>$, where $\{m_1,\cdots,m_r\}$ is a minimal generating system pf $\mathfrak{a}$ Then $m_i\notin(x)$ for $i=1,2,\cdots,r$, because that if so, then there exists an element $y\in R$ such that $a_im_i=yx$ and thus ξ is a zero divisor in $\mathfrak{m}/\mathfrak{a}$.

Therefore, we have $\eta(m_i)=\bar{m}_i\neq 0$ for $i=1,2,\cdots,r$.

It is clear that $\bar{\mathfrak{a}}=<\bar{m}_1,\bar{m}_2,\cdots,\bar{m}_r>$. Thus, it remains to prove that $\{\bar{m}_1,\bar{m}_2,\cdots,\bar{m}_r\}$ is a minimal generating system of $\bar{\mathfrak{a}}$.

We suppose that there are elements $a_1,a_2,\cdots,a_{i-1},a_{i+1},\cdots,a_r$ in R such that

$$\bar{m}_i=\bar{a}_1\bar{m}_1+\cdots+\bar{a}_{i-1}\bar{m}_{i-1}+\bar{a}_{i+1}\bar{m}_{i+1}+\cdots+\bar{a}_r\bar{m}_r.$$

This means that

$$m_i+(x)=a_1m_1+\cdots+a_{i-1}m_{i-1}+a_{i+1}m_{i+1}+\cdots+a_rm_r+(x),$$

and thus there exists an element $a\in R$ satisfying

$$m_i=a_im_i+\cdots+a_{i-1}m_{i-1}+a_{i-1}m_{i-1}+\cdots+a_rm_r+ax.$$

Since

$$m_i-(a_im_i+\cdots+a_{i-1}m_{i-1}+a_{i+1}m_{i+1}+\cdots+a_rm_r)\in\mathfrak{a},$$

we have $a \in \mathfrak{a}$. Therefore, there exist elements $\ell_1, \cdots, \ell_r$ in R such that

$$a = \ell_1 m_1 + \cdots + \ell_i m_i + \cdots + \ell_r m_r.$$

In consequence, we have

$$m_i - (a_1 m_1 + \cdots + a_{i-1} m_{i-1} + a_{i+1} m_{i+1} + \cdots + a_r m_r) = \ell_1 m_1 x + \cdots + \ell_r m_r x$$

and thus $m_i(1 - \ell_i x) = \sum_{\substack{k=1 \\ k \neq i}}^{r} (a_k + \ell_k x) m_k$.

Since $1 - \ell x$ is a unit in R, $m_i = \sum_{\substack{k=1 \\ k \neq i}}^{r} \frac{1}{(1 - \ell_i x)} (a_k + \ell_k x) m_k$. We have a contradition.

Therefore, $\{\bar{m}_1, \cdots, \bar{m}_r\}$ is a minimal generating system of $\bar{\mathfrak{a}}$. That is, $\mu(\bar{\mathfrak{a}}) = \mu(\bar{\mathfrak{a}})$. ///

§4. Projective Modules

A R-module M is called a *projective* R-module if there is a R-module M' such that $M \oplus M'$ is a free R-module (R is a ring). Sometimes, M is called *locally free* if $M_{\mathfrak{m}}$ is afree $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R. From the above definitions about projective modules, We easily see the following facts ([2],[7],[10]).

Property 4.1. (i) For any multiplicative closed subsets S of R, if M is a projective R-module, then M_S is a projective R_S-module.

(ii) If M is a projective R-module and $P \subset R$ is a subring such that R is a free P-module, then M is also a projective P-module.

(iii) If M is a locally free R-module, then $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in$ Spce (R).

Let S be a multiplicative closed subset of R and let M be a R-module.

Definition 4.2. If $U \subset M$ is a submodule of M, then we define such that

$$S(U) = \{m \in M \mid \text{there is an } s \in S \text{ with } sm \in U\}$$

which is called the *S-component* of U. We also define such that

$$\mathscr{U}(M_S) = \text{the set of all submodules of the } R_S\text{-module } M_S$$

and

$$\mathfrak{A}_S(M)=\{U\subset M \text{ (submodule)} \mid S(U)=U\}.$$

It is obvious that

① $U\subset S(U)$, ② $S(S(U))=S(U)$, and

③ $S(\bigcap_{j=1}^{n} U_i)=\bigcap_{i=1}^{n} S(U_i)$ where $U_i (i=1,2,\cdots,n)$ is a submodule of M.

Proposition 4.3. We have the inclusion-preserving bijective mapping

$$\alpha : \mathfrak{A}_S(M)\longrightarrow \mathfrak{A}(M_S) \quad (U\longrightarrow U_S).$$

The inverse mapping asigns to each $U'\in\mathfrak{A}(M_S)$ the submodule $i^{-1}(U')$, where $i : M\longrightarrow M_S$ is the cannonical mapping.

Proof. For each $U'\in\mathfrak{A}(M_S)$ (i.e., U' is a submodule of M_S), let $U=i^{-1}(U')$. If $m\in S(U)$ (i.e., there is an $s\in S$ with $sm\in U$), then $i(sm)=\frac{s}{1}i(m)\in U'$, Thus

$$\frac{1}{s}\cdot\frac{s}{1}i(m)=i(m)\in U'$$

because $\frac{1}{s}\in R_S$. This means that $m\in U$. It follows that $U=S(U)$ and $U\in\mathfrak{A}_S(M)$. Therefore $\alpha^{-1}(U')=i^{-1}(U')=U$ is well-defined.

Moreover, it clear that $U_S\subset U'$ because that for each $\frac{m}{s}(m\in U, s\in S)\in U_S$ $\frac{1}{s}i(m)=\frac{m}{s}\in U'$. For each element $\frac{m}{s}\in U'$, $\frac{m}{1}\in U'$, and so that $m\in U$, Hence, $\frac{m}{s}\in U_S$ It follows that $U'=U_S$. ///

With the above notations, we can easily prove the followings.

(a) For $\mathfrak{p}\in\operatorname{Spec}(R)$,

$$S(\mathfrak{p})=\{r\in R\mid \text{there is an } s\in S \text{ with } sr\in\mathfrak{p}\}=\begin{cases}\mathfrak{p} & \text{if } \mathfrak{p}\cap S=\phi,\\ R & \text{if } \mathfrak{p}\cap S\neq\phi.\end{cases}$$

(b) For an ideal $\mathfrak{a}$ of R such that $\mathfrak{a}=\bigcap_{i=1}^{n}\mathfrak{p}_i$ with $\mathfrak{p}_i\in\operatorname{Spec}(R)$, we have

$$S(\mathfrak{a})=\bigcap_{\mathfrak{p}_i\cap S=\phi}\mathfrak{p}_i,$$

because that

$$S(\mathfrak{a})=\bigcap_{i=1}^{n}S(\mathfrak{p}_i)=\bigcap_{\mathfrak{p}_i\cap S=\phi}S(\mathfrak{p}_i)=\bigcap_{\mathfrak{p}_i\cap S=\phi}\mathfrak{p}_i$$

by (a).

Lemma 4.4. Let $i:R\longrightarrow R_S$ be the cannonical mapping and $\Sigma=\{\mathfrak{p}\in\mathrm{Spec}\,(R)\mid \mathfrak{p}\cap S=\emptyset\}$, where S is a multiplicative closed subset of R.

(i) Every $\beta\in\mathrm{Spec}\,(R_S)$ is of the form $\beta=\mathfrak{p}_S$ with a uniquely determined $\mathfrak{p}$.

(ii) If R is a unique factorization domain (a factorial ring) and $0\notin S$, then R_S is also a unique factorization domain.

Proof. (i) By Proposition 4.3,

$$\alpha:\mathscr{U}_S(R)\longrightarrow\mathscr{U}(R_S)$$

is a bijective mapping. By (a) above, if $\mathfrak{p}\cap S=\emptyset$, then $\mathfrak{p}\in\mathscr{U}_S(R)$. Therefore, it follow that $\Sigma\subset\mathscr{U}_S(R)$.

In particular, in the cannonical mapping

$$i:R\longrightarrow R_S,$$

$i^{-1}(\beta)=\mathfrak{p}$ is a pime ideal of R and $\mathfrak{p}_S=\beta$. In this case, $\mathfrak{p}\in\Sigma$ and by the bijectivity of α such a prime ideal $\mathfrak{p}\in\Sigma$ is determined uniquely.

(ii) By our assumption, R is an integral domain. Hence $i:R\longrightarrow R_S$ is a monomorphism. Since $0\notin S$, if π is a prime element of R and $(\pi)\cap S=\phi$, then $\frac{\pi}{1}$ is a prime element of R_S. If $(\pi)\cap S\neq 0$, then there is an element $a\in R$ such tha $a\pi\in S$. Thus

$$a\cdot\frac{1}{a\pi}=\frac{1}{\pi}\in R_S,$$

and hence $\frac{\pi}{1}$ is a unit in R_S. It follows that every element $\neq 0$ in R_S is either a unit or a product of prime elements.

That is, R_S is a unique factorization domain. ///

Definition 4.5. For a finitely generated projective R-module P and $\mathfrak{p}\in\mathrm{Spec}\,(R)$, $\mu_\mathfrak{p}(P)$ is called the *rank* of P at $\mathfrak{p}$. P is called of rank r if $\mu_\mathfrak{p}(P)=r$ for all $\mathfrak{p}\in\mathrm{Spec}\,(R)$.

Lemma 4.6. For a finitely generated projective R-module P with rank r at $\mathfrak{p}\in$

Spec (R), there exists an element $f\in R\setminus\mathfrak{p}$ such that P_f is a free R_f-module with rank r.

Proof. Let $\{\omega_1, \cdots, \omega_r\}$ be a minimal generating system of the $R_\mathfrak{p}$-module $P_\mathfrak{p}$. We can choose the ω_i as images of elements ω_i^* of P, for if we multiply the ω_i by a common denominator, we again get a minimal generating system of $P_\mathfrak{p}$.

Consider the exact sequence

$$0 \longrightarrow K \longrightarrow R^r \xrightarrow{\alpha} P \longrightarrow C \longrightarrow 0,$$

where $\alpha(e_i)=\omega_i^*$ $(e_i=(0,0,\cdots,1,0,\cdots,0)\in R^r)$, Ker $\alpha=K$ and $C=$coker α. By our hypothesis, $P_\mathfrak{p}=(R_\mathfrak{p})^r$ and $K_\mathfrak{p}=0=C_\mathfrak{p}$. By (i) of Proposition 2.2, we have an element $f\in R\setminus\mathfrak{p}$ such that $C_f=\{0\}$. By (i) of Property 4.1, P_f is a projective R_f-module, and thus

$$0 \longrightarrow K_f \longrightarrow R_f{}^r \longrightarrow P_f \longrightarrow 0$$

is a split squence. That is, $R_f{}^r\cong K_f\oplus P_f$, and thus K_f is a finitely generated R_f-module. As $K_\mathfrak{p}=\{0\}$, by (i) of Proposition 2.2, there exists an $g\in R\setminus\mathfrak{p}$ such that $K_g=\{0\}$. Thu s$(K_f)_g=(K_g)_f=K_{fg}=\{0\}$. Therefore, we have

$$(R_{fg})^r\cong P_{fg}. \ ///$$

Theorem 4.7. Let R be a semi-local domain with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_t$ such that each local ring $R_{\mathfrak{m}_i}$ is a principal ideal domain for $i=1,2,\cdots,t$. Then all finitely generated projective $R[X_1, \cdots, X_n]$-modules are free.

Proof. Our proof is divided into three steps.

(1) We claim that R is a principal ideal domain. Since R is a semi-local ring, we have $J(R)=\text{Max }(R)$ and

$$\dim J(R)=\dim \text{Max }(R)=0.$$

By Corollary 3.6, for

$$u=\text{Max }\{\mu_\mathfrak{p}(\mathfrak{a})+\dim \mathfrak{B}(\mathfrak{p})\,|\,\mathfrak{p}\in\mathfrak{B}(\mathfrak{a})\},$$

$\mu(\mathfrak{a})\leq\text{Max }\{u, d+1\}$, where $\mathfrak{a}$ is an ideal of $R, \mathfrak{p}\in J(R)$ and $\dim J(R)=d$. Thus $\mu(\mathfrak{a})=\text{Max }\{u, 1\}$.

On the other hand $\mathfrak{p}\in J(R)$ implies that $\mathfrak{p}$ is a maximal ideal of R because $J(R)=$ Max (R). By our hypothesis, if $\mathfrak{a}$ is a nonzero ideal, then $\mu_{\mathfrak{p}}(\mathfrak{a})=1$.

Therefore $\mu(\mathfrak{a})=1$ and R is a principal ideal domain.

(2) $n=0$. Then, by (1), R is a principal ideal domain, and since every finitely generated projective module over a principal ideal domain is a free module by Lemma 2.1, our assertion is correct.

(3) $n>0$. We suppose that our theorem was proved for all positive integers $\leq n-1$. We stall put such that

$$S=\text{the multiplicative closed set of all monic polynomials in } R[X_1].$$

We assume that M is a finitely generated projective $R[X_1,\cdots,X_n]$-module. Then M_S is also a finitely generated projective $R[X_1,X_2,\cdots,X_n]_S$-module. Note that $R[X_1,X_2,\cdots,X_n]_S=R[X_1]_S[X_2,\cdots,X_n]$. Hence, if we can prove that $R[X_1]_S$ is a principal ideal domain, then by our inductive hypothesis M_S is a free $R[X_1,\cdots,X_n]_S$-module with finite rank. By Lemma 4.6, there is an element $f\in S$ such that M_f is a free $R[X_1,X_2,\cdots,X_n]_f$-module. By Theorem 1.1 (Quillen-Suslin), M is a free $R[X_1,\cdots,X_n]$-module.

Therefore we have to prove that $R[X_1]_S$ is a principal ideal domain. Since $R[X_1]$ is a factorial ring (Note that every principal ideal domain is a factorial ring), by (ii) of Lemma 4.4 $R[X_1]_S$ is also a factorial ring. We put $K=R[X_1]_S$.

For $\mathfrak{p}\in$Spec (K) with $\mathfrak{p}\cap R=\{0\}$, $K_{\mathfrak{p}}$ is a ring consisting of fractions in $Q(R)[X_1]$ where $Q(R)$ is the quotient field of R. Thus the height of $\mathfrak{p}$ is zero or one, and hence $\mathfrak{p}$ is a principal ideal.

Next, we assume that $\mathfrak{p}\cap R\neq\{0\}$. Since R is a principal ideal domain (and thus R is a factorial ring), there exists a prime element $p\in R$ such that $\mathfrak{p}\cap R=(p)$. Then $K/pK=R/(p)(X_1)$ is a field, hence pK is a maximal ideal of K. In particular, $pK=\mathfrak{p}$, and thus $\mathfrak{p}$ is a principal ideal of K. In consequence, any $\mathfrak{p}\neq(0)$ is therefore generated by a prime element π of K.

From this, we can prove that $K=R[X_1]_S$ is a principal ideal domain as follows.

For $a_1,a_2\in K\setminus\{0\}$, we assume that c is the greatest common divisor of a_1 and a_2. For $\mathfrak{p}=(\pi)\in$Spec (K), we auuume that $\pi^{v_1}|a_1$ and $\pi^{v_2}|a_2$. Then, for $v=$Min $\{v_1,v_2\}$, $\pi^v|c$. Hence, it follows that $(a_1,a_2)\ K_{\mathfrak{p}}=(c)K_{\mathfrak{p}}$ for all $\mathfrak{p}\in$Spec (K). Thus, for every maximal ideal $\mathfrak{m}$ of K,

$$(a_1, a_2)\ K_{\mathfrak{m}}=(c)K_{\mathfrak{m}},\ \text{i.e,}\ (a_1, a_2)_{\mathfrak{m}}=(c)_{\mathfrak{m}}.$$

By Lemma 2.4, we have $(a_1, a_2)=(c)$. ///

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