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REGULAR COVERINGS AND MANIFOLD CRYSTALLIZATIONS

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1. Introduction

Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote the set of vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the neighborhood of a vertex v. By $|X|$, we denote the cardinality of a finite set X. Let $A(G)$ be the *adjacency matrix* of G. Then the *characteristic polynomial* of G is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$. We denote the characteristic polynomial of G by $\Phi(G;\lambda)$. A zero of $\Phi(G;\lambda)$ is called an *eigenvalue* of G. Let $\mathbf{C}$ denote the field of complex numbers, and let D be a digraph. A *weighted digraph* is a pair $D_\omega = (D, \omega)$, where $\omega : E(D) \to \mathbf{C}$ is a function on the set $E(D)$ of edges in D. We call D the *underlying digraph* of D_ω and ω the *weight function* of D_ω. Given any weighted digraph D_ω, the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij} = \begin{cases} \omega(v_iv_j) & \text{if } v_iv_j \in E(D), \\ 0 & \text{otherwise,} \end{cases}$$

and its characteristic polynomial is that of its adjacency matrix. We shall denote the characteristic polynomial of D_ω by $\Phi(D_\omega;\lambda)$.

Let $\vec{G}$ be the digraph obtained by replacing each edge e of G with a pair of oppositely directed edges, say e^+ and e^-. We denote the set of directed edges of $\vec{G}$ by $E(\vec{G})$. Note that the adjacency matrix of the graph G is the same as that of the digraph $\vec{G}$.

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A graph $\tilde{G}$ is called a *covering* of G with the projection $p : \tilde{G} \longrightarrow G$ if there is a surjection $p : V(\tilde{G}) \longrightarrow V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \longrightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We say that $\tilde{G}$ is an *n-fold covering* of G if the covering projection $p|_{V(\tilde{G})}$ is n-to-one.

2. Regular coverings

A covering $p : \widetilde{G} \to G$ is said to be *regular* if there is a subgroup A of the automorphism group $\mathrm{Aut}(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ such that $\widetilde{G}/A$ is isomorphic to G.

By e^{-1} we mean the reverse edge to an edge $e \in E(\vec{G})$. Let Γ be a finite group. A Γ-*voltage assignment* of G is a set function ϕ from the set $E(\vec{G})$ to the group Γ such that $\phi(e^{-1}) = (\phi(e))^{-1}$ for all e in $E(\vec{G})$. The values of ϕ are called Γ-voltages and Γ is called the voltage group.

The *voltage covering graph* $G \times_\phi \Gamma$ derived from $\phi : E(\vec{G}) \to \Gamma$ has as its vertex set $V(G) \times \Gamma$ and as its edge set $E(G) \times \Gamma$, so that an edge of $G \times_\phi \Gamma$ joins a vertex (u, g) to $(v, \phi(e)g)$, for $e = uv \in E(G)$ and $g \in \Gamma$. A vertex (u, g) is denoted by u_g, and an edge (e, g) by e_g. The voltage group Γ acts on $G \times_\phi \Gamma$ as follows: for every $g \in \Gamma$, let $\Phi_g : G \times_\phi \Gamma \to G \times_\phi \Gamma$ denote the graph automorphism defined by $\Phi_g(v_{g'}) = v_{g'g^{-1}}$ on verteces and $\Phi_g(e_{g'}) = e_{g'g^{-1}}$ on edges. Then the natural map $G \times_\phi \Gamma \to (G \times_\phi \Gamma)/\Gamma \cong G$ is a regular $|\Gamma|$-fold covering projection. Gross and Tucker [5] showed that every regular covering of G arises from some voltage assignment of G.

From now on, we consider the voltage group Γ as a finite abelian group. Note that Γ is isomorphic to $\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_\ell}$. For all $\alpha = 1, \cdots, \ell$, let ρ_α denote the generater of $\mathbf{Z}_{n_\alpha}$ so that $\mathbf{Z}_{n_\alpha} = \{\rho_\alpha^0, \rho_\alpha^1, \cdots, \rho_\alpha^{n_\alpha - 1}\}$.

Let ϕ be an Γ-voltage assignment of G. For each $\gamma \in \Gamma$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$, so that the digraph $\vec{G}$ is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\phi,\gamma)}, \gamma \in \Gamma$. Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$. We define an order relation $\leq$ on $X \times Y$ as follows: for any two vertices (x_i, y_p) and (x_j, y_q) of $X \times Y$, $(x_i, y_p) \leq (x_j, y_q)$ if and only if either $p < q$ or $p = q$ and $i \leq j$. From now on, we assume that the order relation on a product of two ordered sets defined in this way. We define an order relation $\leq$ on $\mathbf{Z}_{n_\alpha}$ by $\rho^\ell \leq \rho^m$ if and only if $\ell \leq m$. This order relation gives an order relation on Γ. For any $\gamma \in \Gamma$, let $P(\gamma)$ be the ***permutation matrix associated with*** γ under the above order. We note that the set of vertices of $G \times_\phi \Gamma$ also has the corresponding order relation if an order relation on $V(G)$ is given.

To find the adjacency matrix $A(G \times_\phi \Gamma)$ according to this order relation, we note that an edge of the covering $G \times_\phi \Gamma$ joining vertices (u, γ') and (v, γ'') gives entry 1 in the matrix $A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma)$ if $uv \in E(\vec{G})$, $\gamma'' = \phi(uv)\gamma'$ and $\phi(uv) = \gamma$, where $\otimes$ means the tensor product of two matrices. Note that the $(m,n)^{\text{th}}$ block of $A \otimes B$ is Ab_{mn}, where b_{mn} is the (m,n) $^{\text{th}}$ entry of B. Clearly, for each $\alpha = 1, \cdots, \ell$, the permutation matrix $P(\rho_\alpha)$ associated with ρ_α is the $n_\alpha \times n_\alpha$ matrix

$$\begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

and it is similar to

$$D(\rho_\alpha) \stackrel{\text{def}}{=} \begin{bmatrix} 1 & & & & \\ & \zeta_\alpha & & 0 & \\ & & \zeta_\alpha^2 & & \\ & 0 & & \ddots & \\ & & & & \zeta_\alpha^{n_\alpha - 1} \end{bmatrix},$$

where $\zeta_\alpha = \exp(\frac{2\pi}{n_\alpha} i)$ for $1 \leq \alpha \leq \ell$.

Let $\gamma \in \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_\ell}$. Then $\gamma = (\rho_1^{k_1}, \rho_2^{k_2}, \cdots, \rho_\ell^{k_\ell})$. By the virtue of properties of the tensor product of matrices and the given order relation on Γ, we have

$$P(\gamma) = \left(\begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix} \right)^{k_1} \otimes \cdots \otimes \left(\begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix} \right)^{k_\ell},$$

which is similar to $D(\rho_1)^{k_1} \otimes \cdots D(\rho_\ell)^{k_\ell}$. We summarize our discussion as follow.

THEOREM 1. *Let $\Gamma = \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_\ell}$ and let ϕ be a Γ-voltage assignment of G. Then, the adjacency matrix of a regular covering graph $G \times_\phi \Gamma$ is*

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\vec{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right) \otimes P(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}),$$

where $P(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell})$ is the permutation matrix associated with $(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell})$ and the adjacency matrix is similar to

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\vec{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right) \otimes \left(D(\rho_1)^{k_1} \otimes \cdots D(\rho_\ell)^{k_\ell}\right).$$

Let ϕ be a Γ-voltage assignment of G. For each $(s_1,\cdots,s_\ell)$ with $0 \le s_\alpha < n_\alpha$ and $1 \le \alpha \le \ell$, we define a weight function $\omega_{(s_1,\cdots,s_\ell)}(\phi) : E(\vec{G}) \to \mathbf{C}$ by

$$\omega_{(s_1,\cdots,s_\ell)}(\phi)(e) = \prod_{\alpha=1}^{\ell} \left(\zeta_\alpha^{k_\alpha}\right)^{s_\alpha} \text{ for } \phi(e) = \prod_{\alpha=1}^{\ell} \rho_\alpha^{k_\alpha}.$$

Note that the nonzero blocks of the matrix

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\vec{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right) \otimes \left(D(\rho_1)^{k_1} \otimes \cdots D(\rho_\ell)^{k_\ell}\right)$$

are diagonal ones and its $(o(s_1,\cdots,s_\ell), o(s_1,\cdots,s_\ell))$-th diagonal block is

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\vec{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right) (\zeta_1^{k_1})^{s_1}(\zeta_2^{k_2})^{s_2}\cdots(\zeta_\ell^{k_\ell})^{s_\ell},$$

where $o(s_1,\cdots,s_\ell) = s_\ell(n_1 n_2 \cdots n_{\ell-1}) + s_{\ell-1}(n_1 n_2 \cdots n_{\ell-2}) + \cdots + s_1 + 1$. Thus, we have

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\vec{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right) \otimes \left(D(\rho_1)^{k_1} \otimes \cdots D(\rho_\ell)^{k_\ell}\right) = \bigoplus_{(s_1,\cdots,s_\ell)} A\left(\vec{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)}\right).$$

We summarize our discussion as follow.

THEOREM 2.

$$\Phi(G\times_\phi\Gamma;\lambda)=\prod_{(s_1,\cdots,s_\ell)}\Phi\left(\vec{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};\lambda\right).$$

Now, we introduce a calculation for the characteristic polynomial $\Phi\left(\vec{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};\lambda\right)$. To do this, we introduce some notations. An undirected simple graph S is call a *basic figure* if each component of S is either a cycle C_q $(q=1,3,4,\cdots)$ or K_2. For a basic figure S, let $\kappa(S)$ be the number of components of S and $\mathcal{C}(S)$ be the set of all cycles in S. Let $\mathcal{G}_j(G)$ be the set of all subgraphs of G which are basic figures with j vertices. Note that every cycle C in G induces two directed cycles in $\vec{G}$, say C^+ and C^-. Then, the following theorem can be found in [9] (Theorem 5)

THEOREM 3. *Let Γ be a finite abelian group and let ϕ be a Γ-voltage assignment of G. Let ω be one of $\omega_{(s_1,\cdots,s_\ell)}(\phi)$. Then, we have*

$$\Phi\left(\vec{G}_\omega;\lambda\right)=\lambda^{|V(G)|}+\sum_{j=1}^{|V(G)|}\left(\sum_{S\in\mathcal{G}_j(G)}(-1)^{\kappa(S)}2^{|\mathcal{C}(S)|}\prod_{C\in\mathcal{C}(S)}\mathbf{R}e\left(\omega(C^+)\right)\right)\lambda^{|V(G)|-j},$$

where $\mathbf{R}e\left(\omega(C^+)\right)$ is the real part of $\prod_{e\in C^+}\omega(e)$. Moreover, if $\phi(e)$ is of order 2 for each $e\in E(\vec{G})$, then

$$\Phi(\vec{G}_\omega;\lambda)=\lambda^{|V(G)|}+\sum_{j=1}^{|V(G)|}\left(\sum_{S\in\mathcal{G}_j(G)}(-1)^{\kappa(S)+\mathcal{N}_\omega(S)}2^{|\mathcal{C}(S)|}\right)\lambda^{|V(G)|-j},$$

where $\mathcal{N}_\omega(S)$ is the number of cycles in S such that $\omega(C^+)=-1$.

Note that the second statement of Theorem 3 is different that of Theorem 5 in [9] but they are equal because every eigenvalues of any permutation matrix associted with an element of order 2 is either 1 or -1.

We also note that Theorem and Theorem 2 are true for pseudographs, but Theorem 3 is not.

Now, we give a theorem which is the pseudograph version for Theorem 3. In an undirected pseudograph, two basic figures S_1 and S_2 are equivalent if the identity map on the vertex set $V(G)$ induces an isomorphism between S_1 and S_2. We denote the set of equivalence classes of $\mathcal{G}_j(G)$ by $[\mathcal{G}_j(G)]$ for $j = 1, \cdots V(G)$. Let $[S]$ be an element of $[\mathcal{G}_j(G)]$. Then, $[S]$ consists of equivalence classes of K_2 and cycles. Let $\mathcal{E}(K_2[S])$ be the equivalence classes of the copies of K_2 and $\mathcal{E}(\mathcal{C}[S])$ the equivalence classes of the cycles in $[S]$. Note that every copies of K_2 in G induces two directed edges in $\vec{G}$, say e^+ and e^-, and every loop is a cycle of length 1. Then, by using a method similar to the proof of Theorem 5 in [9], we can prove the following theorem.

THEOREM 4. *Let Γ be a finite abelian group and let ϕ be a Γ-voltage assignment of G. Let ω be one of $\omega_{(s_1,\cdots,s_\ell)}(\phi)$. Then, for each $[S] \in [\mathcal{G}_j(G)]$, the contribution of $[S]$ in the the coefficent of $\lambda^{|V(G)|-j}$ of $\Phi\left(\vec{G}_\omega;\lambda\right)$ is*

$$(-1)^{\kappa(S)} \prod_{[e]\in\mathcal{E}(K_2[S])} \left(\sum_{e\in[e]} \omega(e^+)\right)\left(\sum_{e\in[e]} (\omega(e^+))^{-1}\right) 2^{|\mathcal{E}(\mathcal{C}[S])|} \prod_{[C]\in\mathcal{E}(\mathcal{C}[S])} \left(\sum_{C\in[C]} \mathbf{Re}\left(\omega(C^+)\right)\right),$$

where $\mathbf{Re}\left(\omega(C^+)\right)$ is the real part of $\prod_{e\in C^+} \omega(e)$ and S is a representative of $[S]$.

3. Applications

If G is a bipartite graph, then $\mathcal{G}_j(G) = \emptyset$ for $1 \le j = \text{odd} \le |V(G)|$. By theorem 3,

$$\Phi(\vec{G}_\omega; -\lambda) = (-1)^{|V(G)|}\Phi(\vec{G}_\omega;\lambda),$$

where ω is of the form $\omega_{(s_1,\cdots,s_\ell)}(\phi)$. Thus for any Γ-voltage assignment ϕ, we have

$$\Phi(G\times_\phi\Gamma;-\lambda) = (-1)^{|V(G)|\,|\Gamma|}\Phi(G\times_\phi\Gamma;\lambda) = (-1)^{|V(G\times_\phi\Gamma)|}\Phi(G\times_\phi\Gamma;\lambda).$$

Because a graph is bipartite if and only if $\Phi(G;-\lambda) = (-1)^{|V(G)|}\Phi(G;\lambda)$, we have the following corollary.

COROLLARY 1. *Let G be a bipartite graph and let ϕ be a Γ-voltage assignment of G. Then the regular covering graph $G\times_\phi\Gamma$ is also bipartite.*

Let G be a k-regular graph and let $\tau(G)$ be the number of spanning trees of G. Then $\tau(G) = |V(G)|^{-1}\Phi'(G;k)$, where Φ' denotes the derivative of the characteristic polynomial $\Phi(G;\lambda)$ ([1], p. 36). It is known that $\Phi(G;k) = 0$ for a k-regular graph G. Thus we have,

COROLLARY 2. *Let G be a k-regular connected graph and let ϕ be a Γ-voltage assignment of G. Then,*

(1) $G \times_\phi \Gamma$ is connected if and only if $\Phi(\vec{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};k) \neq 0$ for all $(s_1,\cdots,s_\ell) \neq (0,\cdots,0)$.

(2) $\tau(G \times_\phi \Gamma) = \frac{1}{|\Gamma|}\tau(G) \prod_{(s_1,\cdots,s_\ell)\neq(0,\cdots,0)} \Phi\left(\vec{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};k\right).$

The platonic graphs are the graphs whose vertices and edges are the vertices and the esges of the platonic solids. They named tetrahedron, cube, octahedron, dodecahedron, and icosahedron. We calculate the characteristic polynomials of the platonic graphs as examples of our discussions. The characteristic polynomial of tetrahedron $\mathcal{T}$ is well known because it is isomorphic to the complete graph K_4. It is also known that the cube $\mathcal{Q}$ is the double covering of K_4 which corresponded the voltage assignment whose values of all edges are the nontrivial element in $\mathbf{Z}_2$. Thus

$$\Phi(\mathcal{Q};\lambda) = \Phi(K_4;\lambda)(-1)^4\Phi(K_4;-\lambda) = (x-3)(x+1)^3(x+3)(x-1)^3.$$

Now, we describe the remaining platonic graphs as a regular covering graph of some graphs.

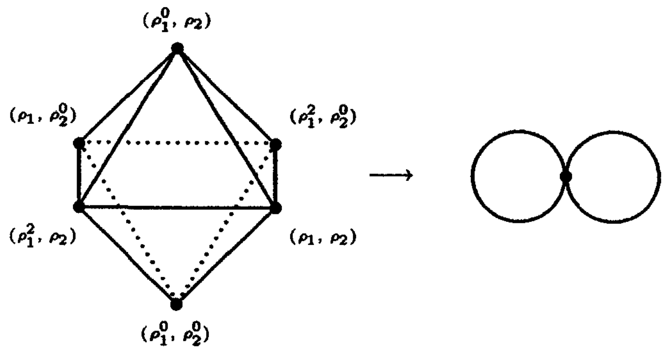

Figure 1. The octahedron covers the figure eight graph

Figure 1 shows that the octahedron $\mathcal{O}$ is a 6-fold regular covering graph of the figure eight graph $\mathcal{F}$ and the covering projection is that the outer cycle of length 6 covers the right loop of $\mathcal{F}$ and the two cycles of length 3 in $\mathcal{O}$ covers the left loop of $\mathcal{F}$. Note that the corresponding $\mathbf{Z}_3 \times \mathbf{Z}_2$-voltage assignment ϕ of $\mathcal{F}$ is the map which assigns (ρ_1, ρ_2^0) on the left loop with clockwise direction and (ρ_1, ρ_2) on the right loop with counterclockwise direction. From now on, we consider that all loops are directed counterclockwise and all edges are directed from the lower labeled vertex to the higher labeled vertex in the base graph.

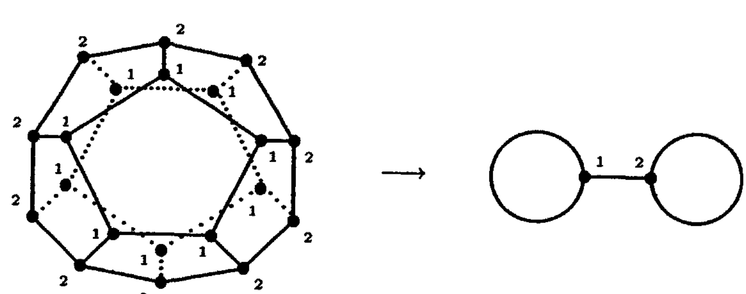

Figure 2. The dodecahedron covers the dumbbell graph

Figure 2 shows that the dodecahedron $\mathcal{D}$ covers the dumbbell graph and the covering projection is the graph homomorphism which preserves the labelling and the corresponding $\mathbf{Z}_5 \times \mathbf{Z}_2$-voltage assignment ϕ on the dumbbell graph $\mathcal{M}$ is the map which assigns (ρ_1, ρ_2^0) on the left loop, (ρ_1^3, ρ_2) on the right loop and the trival member on the edge between the vertex 1 and the vertex 2. We describe the icosahedron $\mathcal{I}$ as a regular covering graph of the graph $\mathcal{H}$ which drawn in Fig 3.

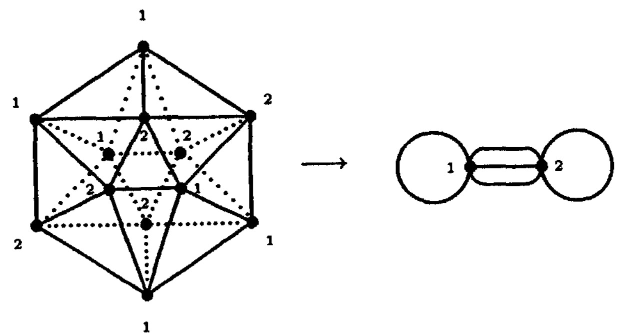

Figure 3. The icosahedron covers the graph $\mathcal{H}$

In Figure 3, the covering projection is the graph homomorphism which preserves the labelling and the corresponding $\mathbf{Z}_3 \times \mathbf{Z}_2$-voltage assignment ϕ of $\mathcal{H}$ is the map which assigns the member $(\rho_1,\ \rho_2)$ on the left loop, the $(\rho_1,\ \rho_2^0)$ on the right loop, and the members $(\rho_1^0,\ \rho_2)$, $(\rho_1,\ \rho_2^0)$ and $(\rho_1^2,\ \rho_2)$ on the three edges between the vertices 1 and the vertex 2. We calculate the characteristic polynomial of Dodecahedron $\mathcal{D}$ as an example. Let $\zeta = \exp(\frac{2\pi}{5}i)$. Then, by Theorem 4,

$$\Phi\left(\vec{\mathcal{M}}_{\omega(s_1, s_2)(\phi)}; \lambda\right) = \lambda^2 - \left(\left(\zeta^{s_1} + \zeta^{-s_1}\right) + (-1)^{s_2}\left(\zeta^{3s_1} + \zeta^{-3s_1}\right)\right)\lambda + (-1)^{s_2}\left(\zeta^{2s_1} + \zeta^{-2s_1} + \zeta^{4s_1} + \zeta^{-4s_1}\right) - 1,$$

and is equal to

$$\begin{cases} (\lambda-3)(\lambda-1) & \text{for } s_1 = s_2 = 0, \\ (\lambda+2)(\lambda-1) & \text{for } s_1 \neq 0, s_2 = 0, \\ (\lambda^2 - 5) & \text{for } s_1 = 0, s_2 = 1, \\ \lambda\left(\lambda - \left(2\left(\zeta^{s_1} + \zeta^{-s_2}\right) + 1\right)\right) & \text{for } s_1 \neq 0, s_2 = 1. \end{cases}$$

By using Theorem 2, we have

$$\Phi(\mathcal{D};\lambda) = (\lambda-3)(\lambda^2-5)^3(\lambda-1)^5\,\lambda^4\,(\lambda+2)^4.$$

The characteristic polynomials of platonic graphs and the number of spanning trees of them are given in the following table.

	$\Phi(G;\lambda)$	$\tau(G)$
$\mathcal{T}$	$(x-3)(x+1)^3$	16
$\mathcal{Q}$	$(x-3)(x-1)^3(x+1)^3(x+3)$	384
$\mathcal{O}$	$(\lambda-4)\lambda^3(\lambda+2)^2$	384
$\mathcal{D}$	$(\lambda-3)(\lambda^2-5)^3(\lambda-1)^5\lambda^4(\lambda+2)^4$	5,184,000
$\mathcal{I}$	$(\lambda-5)(\lambda+1)^5(\lambda^2-5)^3$	5,184,000

As the final applications of our results, we calculate the characteristic polynimial of the generalized Petersen graph $P(n,k)$, which consists of an outer n-cycle, n spokes incident to the vertices of this n-cycle, and an inner n-cycle attached by joining its vertices to every k-th spoke. Let ρ be a generator of $\mathbf{Z}_n$ and let $\phi(n,k)$ be the $\mathbf{Z}_n$-voltage assignment of the dumbbell graph $\mathcal{M}$ which assigns ρ on the left loop with counterclockwise direction, ρ^k on the right loop with counterclockwise direction, and the identity on the the directed edge whose initial vertex is 1 and terminal vertex is 2. Then, $\mathcal{M}\times_{\phi(n,k)}\mathbf{Z}_n$ is isomorphic to $P(n,k)$. This gives that every generalized Petersen graphs are a regular covering of the dumbbell graph. By Theorem 4,

$$\begin{aligned}
&\Phi(\vec{\mathcal{M}}_{\omega_{(s_1)}(\phi(n,k))};\lambda)\\
&= \lambda^2 - \left(\zeta+\zeta^{n-1}+\zeta^k+\zeta^{n-k}\right)\lambda + \left(\zeta+\zeta^{n-1}\right)\left(\zeta^k+\zeta^{n-k}\right) - 1\\
&= \lambda^2 - 2\left(\cos(\frac{2s_1\pi}{n}) + \cos(\frac{2ks_1\pi}{n})\right)\lambda + 4\cos(\frac{2s_1\pi}{n})\cos(\frac{2ks_1\pi}{n}) - 1,
\end{aligned}$$

where $\zeta = \exp(\frac{2\pi}{n}i)$. This gives the following theorem.

THEOREM 5.

$$\Phi\left(P(n,k);\lambda\right) = (\lambda-3)(\lambda-1)\prod_{h=1}^{n-1}\left(\lambda^2 - 2\left(\cos(\frac{2h\pi}{n}) + \cos(\frac{2kh\pi}{n})\right)\lambda \right.$$
$$\left. +4\cos(\frac{2h\pi}{n})\cos(\frac{2kh\pi}{n}) - 1\right).$$

Theorem 5 gives that $(P(5,2);\lambda) = (\lambda+2)^4(\lambda-1)^5(\lambda-3)$ because $\zeta+\zeta^2+\zeta^3+\zeta^4 = -1$ for $n=5$.

4. Further remarks

Mohar [10] showed that every *PL n-manifold M* can be represented as a branched covering of S^n which is corresponded to the graph H_n consisting of two vertices and $n+1$ parallel edges between them, and the simplicial scheme h_n which sends directed edges of H_n to its inverse. He also showed that the isomorphism classes of d-fold branched coverings of $S^n = K(H_n, h_n)$ and the isomorphism classes of d-fold coverings of H_n are one - to - one correspondence with the conjugacy classes of homomorphisms $\Pi_1(H_n, *) \to S_d$. Kwak and Lee[9] showed that the number of conjugacy classes of homomorphisms $\Pi_1(G,*) \to S_d$ is

$$\sum_{\ell_1+2\ell_2+\cdots d\ell_d=d} \left(\ell_1!2^{\ell_2}\ell_2!\cdots d^{\ell_d}\ell_d!\right)^{\beta(G)-1}$$

where $\beta(G)$ is the betti number of G. It is clear that $\beta(H_n)=n$. Thus the number of PL n-manifold which are d-fold branched covering of $S^n = K(H_n, h_n)$ up to covering isomorphism is bounded by the number of d-fold coverings of H_n up to covering isomorphism and is equal to

$$\sum_{\ell_1+2\ell_2+\cdots d\ell_d=d} \left(\ell_1!2^{\ell_2}\ell_2!\cdots d^{\ell_d}\ell_d!\right)^{n-1}.$$

But, we can not answer the following question.

QUESTION. Are there some relationships between the spectral structures of the coverings of H_n and the structures of corresponding pseudocomplexes (crystallizations)?

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