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ESTIMATES ON MEAN ENTROPY FOR QUASI–FREE STATES ON CAR AND CCR ALGEBRAS

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1. Introduction

The notion of entropy first arose in thermodynamics as a measure of the heat absorbed, or emitted, when extremal work is done on a system. In the subsequent development of classical statistical mechanics this quantity was related to the order, or disorder, of the microscopic particles which constitute the system. The equilibrium states in statistical mechanics have been characterized by a principle of maximum mean entropy at fixed energy[1]. The concept of (mean) entropy was then abstracted from statistical mechanics to dynamical systems via the work of Kolmogorov and Sinai[4,8] and become a key notion in ergodic theory[2].

A quantum or non-commutative analogue of the Kolmogorov - Sinai dynamical entropy was required for both to provide an important mathematical concept for quantum dynamical systems and to be applicable in quantum statistical mechanics. There have been several attempts to generalize the classical theory to non-commutative case. Recently Connes, Narenhofer and Thirring was able to extend the KS dynamical entropy to quantum dynamical systems[5] and their results for AF-algebras have been extended to non-AF algebras[7]. It has also been established that in many situation the dynamical entropy for space translations in quantum spin systems turn out to be the mean entropy[5,6,9].

In this paper we estimate the mean entropy for quasi-free states on algebras of canonical commutation relations (CCR) and of canonical anti-commutation relations (CAR) over $L^2(\mathbf{R}^\nu)$, and derive exact formulae for the entropies. Our ultimate goal will be to show that the dynamical

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entropy for space translation, and mean entropy for the systems are coincide. We leave it to further studies. The CAR and CCR algebras, we are dealing with arise naturally in quantum particle systems and so the estimate of the mean entropy itself would be meaningful.

The contents of the paper are as follows : In Section 2 we briefly review the notion of quasi-free states on CAR and CCR algebras and then state our main result. In Section 3, we investigate local entropies of the systems in details. We establish our main result in Section 4.

2. Quasi-free States on CAR and CCR Algebras and Main Result

In this section we briefly review the notion of quasi-free states on CAR and CCR algebras, and then state our main result. For the detailed descriptions on CAR and CCR algebras, we refer the reader to Ref.[1].

Let us first describe quasi-free states on the algebra of canonical commutation relations (CCR algebra) briefly. For any bounded region $\Lambda \subset \mathbf{R}^\nu$, let $\mathcal{A}_\Lambda$ denote the C^*-algebra generated by non-zero element $W(f)$, $f \in L^2(\Lambda, d^\nu x)$, satisfying

$$W(-f) = W(f)^* \tag{2.1}$$

$$W(f)W(g) = \exp(-iIm(f,g))W(f+g) \tag{2.2}$$

and let $\mathcal{A}$ be the quasi-local algebra defined by the norm closure of $\bigcup_{\Lambda \subset \mathbf{R}^\nu} \mathcal{A}_\Lambda$. Let ω be a regular state on $\mathcal{A}$. Let $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}), \Omega_\omega)$ be the canonical representation of $\mathcal{A}$ with respect to the state ω. For each $f \in L^2(\mathbf{R}^\nu)$ denote the infinitesimal generator of the unitary group $t \mapsto \pi_\omega(W(tf))$ by $\Phi_\omega(f)$. The annihilation and creation operators for each $f \in L^2(R^\nu)$ given by

$$a(f) = \frac{1}{\sqrt{2}}(\Phi_\omega(f) + i\Phi_\omega(if)), \quad a^*(f) = \frac{1}{\sqrt{2}}(\Phi_\omega(f) - i\Phi_\omega(if)) \tag{2.3}$$

are densely defined, closed, $a(f)^* = a^*(f)$ and satisfy the following canonical commutation relation

$$a(f)a^*(g) - a^*(g)a(f) = (f,g)1 \tag{2.4}$$

for any $f, g \in L^2(\mathbf{R}^\nu)$. For any positive $A \in \mathcal{L}(L^2(\mathbf{R}^\nu))$ the quasi-free state ω_A on the CCR algebra over $L^2(\mathbf{R}^\nu)$ is defined by

$$\omega_A(W(f)) = \exp\{-\frac{1}{2}\omega_A(\Phi_\omega(f)^2)\}$$

$$\omega_A(\Phi_\omega(f)^2) = \frac{1}{2}\omega_A\big(a(f)a^*(f) + a^*(f)a(f)\big) \tag{2.5}$$

$$= \frac{1}{2}(f, (1+2A)f)$$

for any $f \in L^2(\mathbf{R}^\nu)$. The relation (2.5) implies that

$$\omega_A\big(a^*(f)a(g)\big) = (g, Af) \tag{2.6}$$

for any $f, g \in L^2(\mathbf{R}^\nu)$ and any higher order truncated functional equal to zero.

The algebra of canonical anti-commutation relations (CAR algebra) over $L^2(\mathbf{R}^\nu)$ is the C^*-algebra generated by the identity 1 and the elements $a(f), f \in L^2(\mathbf{R}^\nu)$, satisfying the following canonical anti-commutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f, g)1 \tag{2.7}$$

for any $f, g \in L^2(\mathbf{R}^\nu)$. For any positive $A \in \mathcal{L}(L^2(\mathbf{R}^\nu))$ the quasi-free state ω_A on the CCR algebra is defined by

$$\omega_A\big(a^*(f)a(g)\big) = (g, Af)$$

for any $f, g \in L^2(\mathbf{R}^\nu)$, and any higher order truncated functionals equal to zero[1,10].

We now describe our result. Let $K : \mathbf{R}^\nu \to \mathbf{R}$ be a (integrable) function and let $\widehat{K}$ be its unconventional Fourier transform:

$$\widehat{K}(k) = \int K(x)\exp(-ikx)d^\nu x \tag{2.8}$$

where $kx = \sum_{i=1}^{\nu} k_i x_i$. Let A be the operator on $L^2(\mathbf{R}^\nu)$ given by

$$(Af)(x) = \int K(x-y)f(y)d^\nu y \tag{2.9}$$

Then the quasi-free state ω_A is translational invariant. Throughout the paper we assume that the following conditions hold:

ASSUMPTION A. The Fourier transform $\widehat{K}$ of K satisfies the following conditions:

(a) For the CAR algebra, $0 \leq \widehat{K}(k) \leq 1$, and for CCR algebra there exist a positive constant M such that $0 \leq \widehat{K}(k) \leq M$.

(b) There exists a positive constant $\alpha(<1)$ and M' such that

$$|\widehat{K}(k)| \leq M' \prod_{i=1}^{\nu} (1+|k_i|)^{-(1+\alpha)}.$$

(c) $\widehat{K}$ be continuous on $\mathbf{R}^\nu$.

REMARKS. (1) Assumption A(a) implies that $0 \leq A \leq 1$ for the CAR algebra and $0 \leq A \leq M1$ for the CCR algebra. Assumption A(c) can be relaxed in some way. However for technical convenience we stick to the condition (c).

(2) The states for ideal Fermi and Bose gases at inverse temperature $\beta = 1/T$ with activity $z = e^{-\mu}$ are the quasi-free state corresponding to

$$\widehat{K}(k) = \frac{\exp(-k^2-\mu)}{1+\exp(-k^2-\mu)} \qquad (CAR) \tag{2.10}$$

and

$$\widehat{K}(k) = \frac{\exp(-k^2-\mu)}{1-\exp(-k^2-\mu)}, \quad \mu > 0, \qquad (CCR) \tag{2.11}$$

respectively. Obviously these satisfy the condition in Assumption A.

THEOREM 2.1. *Let A be the operator on $L^2(R^\nu)$ defined by (2.8) - (2.9). Let $s(\omega_A)$ be the mean entropy for the quasi-free state ω_A. Under Assumption A the following results hold :*

(a) *For the CAR algebra one has*

$$s(\omega_A) = -\frac{1}{(2\pi)^\nu} \int \{\widehat{K}(k)\log \widehat{K}(k) + (1-\widehat{K}(k))\log(1-\widehat{K}(k))\} d^\nu k$$

(b) *For the CCR one has*

$$s(\omega_A) = \frac{1}{(2\pi)^{\frac{\nu}{2}}} \int \{-\widehat{K}(k)\log \widehat{K}(k) + (1+\widehat{K}(k))\log(1+\widehat{K}(k))\} d^\nu k$$

REMARK. The exact definition of the mean entropy will be given in Section 3.

3. Some Properties of Local Entropies

For any bounded region $\Lambda \subset \mathbf{R}^\nu$ let P_Λ denote the projection operator from $L^2(\mathbf{R}^\nu)$ to $L^2(\Lambda)$, and let

$$A_\Lambda \equiv P_\Lambda A P_\Lambda \tag{3.1}$$

For a notational simplification we suppress ω_A in the notation, i.e., $\mathcal{H} = \mathcal{H}_{\omega_A}$, $\Omega = \Omega_{\omega_A}$, etc. Let $\mathcal{H}_\Lambda$ be the subspace of $\mathcal{H}$ spanned by $\pi(\mathcal{A}_\Lambda)\Omega$, and denotes the trace over $\mathcal{H}_\Lambda$ by $Tr_{\mathcal{H}_\Lambda}$. From Assumption A(b) it follows that

$$Tr_{\mathcal{H}_\Lambda}(A_\Lambda) = (2\pi)^{-\nu}|\Lambda| \int \widehat{K}(k) d^\nu k < \infty$$

and so the state ω_A is locally normal [1,10]. Let ρ_Λ be the density operator corresponding to $\omega_\Lambda = \omega_A|_{\mathcal{A}_\Lambda}$. We then have that for $f, g \in L^2(\Lambda)$

$$\begin{aligned} \omega_\Lambda(a^*(f)a(g)) &= Tr_{\mathcal{H}_\Lambda}(\rho_\Lambda a^*(f)a(g)) \\ &= (g, A_\Lambda f) \end{aligned} \tag{3.2}$$

The *local entropy* for the state ω_A is given by

$$S_\Lambda = -Tr_{\mathcal{H}_\Lambda}(\rho_\Lambda \log \rho_\Lambda) \tag{3.3}$$

and the *mean entropy* is defined by

$$s(\omega_A) = \lim_{\Lambda \uparrow R^\nu} \frac{S_\Lambda}{|\Lambda|}. \tag{3.4}$$

The above limit exists by the subadditivity of S_Λ[1,10].

PROPOSITION 3.1. *Let S_Λ be the local entropy defined by (3.3) for each bounded region $\Lambda \subset R^\nu$. One has that*

$$S_\Lambda = -Tr_{L^2(\Lambda)}(A_\Lambda \log A_\Lambda + (1 - A_\Lambda)\log(1 - A_\Lambda)) \qquad (CAR)$$

and

$$S_\Lambda = -Tr_{L^2(\Lambda)}(A_\Lambda \log A_\Lambda - (1 + A_\Lambda)\log(1 + A_\Lambda)) \qquad (CCR)$$

for any $\Lambda \subset R^\nu$. The right hand sides of the above expressions are finite for any bounded region Λ in R^ν.

Proof. We first consider the local entropy of the CCR algebra. Since ω_{A_Λ} is a quasi-free state on $\pi(\mathcal{A}_\Lambda)$, there exists a trace class positive operator B on $L^2(\Lambda)$

$$\rho_\Lambda = \Gamma(B)/Tr(\Gamma(B)) \tag{3.5}$$

where $\Gamma(B)$ is the second quantization of B[1]. We note that

$$\Gamma(B)a^*(f) = a^*(Bf)\Gamma(B). \tag{3.6}$$

Thus it follows from (3.2), (3.5) and (3.6) that for any $f, g \in L^2(\Lambda)$

$$\begin{aligned}\omega_\Lambda(a^*(f)a(g)) &= Tr(\Gamma(B)a^*(f)a(g))/Tr(\Gamma(B)) \\ &= Tr(a(g)a^*(Bf)\Gamma(B))/Tr(\Gamma(B)) \\ &= (g, Bf) + Tr(\Gamma(B)a^*(Bf)a(g))/Tr(\Gamma(B)) \\ &= (g, Bf) + \omega_\Lambda(a^*(Bf)a(g))\end{aligned}$$

Choosing $f = (1-B)^{-1}g$, we have that

$$\omega_\Lambda\big(a^*(g)a(g)\big) = \big(g, \frac{B}{1-B}g\big).$$

From (3.2) and the above result we conclude that $A_\Lambda = B(1-B)^{-1}$ and so

$$B = \frac{A_\Lambda}{1+A_\Lambda} \tag{3.7}$$

Let $\{\lambda_i\}$ be the eigenvalues of B counting multiplicities. Then it can be easy to show that [1]

$$\begin{aligned}Tr_{\mathcal{H}_\Lambda}(\Gamma(B)) &= \sum_{k=1}^{\infty} \sum_{\substack{m_1,\cdots,m_k=0: \\ \{n_1,\cdots,n_k\}\subset N}}^{\infty} (\lambda_{n_1}^{m_1}\cdots\lambda_{n_k}^{m_k}) \\ &= \prod_j (1-\lambda_j)^{-1} \qquad (3.8) \\ &= \exp\{-Tr_{L^2(\Lambda)}\big(\log(1-B)\big)\} \qquad (3.9)\end{aligned}$$

and

$$\begin{aligned}Tr_{\mathcal{H}_\Lambda}(\Gamma(B)\log\Gamma(B)) &= \sum_{k=1}^{\infty} \sum_{\substack{m_1,\cdots,m_k=0: \\ \{n_1,\cdots,n_k\}\subset N}}^{\infty} (\lambda_{n_1}^{m_1}\cdots\lambda_{n_k}^{m_k})\log(\lambda_{n_1}^{m_1}\cdots\lambda_{n_k}^{m_k}) \\ &= \sum_{i=1}^{\infty}\big[\prod_{j\neq i}(1-\lambda_j)^{-1}\big]\big(\sum_{m=0}^{\infty}\lambda_i^m\log\lambda_i^m\big) \\ &= \sum_{i=1}^{\infty}\big[\prod_{j\neq i}(1-\lambda_j)^{-1}\big](\log\lambda_i)\lambda_i(1-\lambda_i)^{-2}\end{aligned}$$

Therefore from (3.8) it follows that

$$
\begin{aligned}
-\frac{1}{Tr(\Gamma(B))}Tr(\Gamma(B)\log\Gamma(B)) &= -\sum_{i=1}^{\infty}\frac{\lambda_i}{1-\lambda_i}\log\lambda_i \\
&= -Tr_{L^2(\Lambda)}\left(\frac{B}{1-B}\log B\right)
\end{aligned}
\tag{3.10}
$$

From (3.3), (3.9) and (3.10) we obtain that

$$
\begin{aligned}
S_\Lambda &= -Tr_{L(\Lambda)}\left(\frac{B}{1-B}\log B\right) - Tr_{L^2(\Lambda)}\log(1-B) \\
&= -Tr(A_\Lambda\log A_\Lambda) + Tr((1+A_\Lambda)\log(1+A_\Lambda))
\end{aligned}
$$

Here we have used (3.7) to derived the second equality. Thus we have derive the expression for the CCR algebra in Proposition 3.1.

Next let us consider the CAR algebra. The equality in the Proposition has been essentially derived in Ref.[10]. For the reader's convenience we sketch the basic idea. Using CAR relations in (2.7) and the method to obtain (3.7) we conclude that there is a positive operator D on $L(\Lambda)$ such that

$$
\begin{aligned}
\rho_\Lambda &= \Gamma(D)/Tr(\Gamma(D)) \\
D &= \frac{A_\Lambda}{1-A_\Lambda}
\end{aligned}
\tag{3.11}
$$

Let $\{\gamma_i\}$ be the eigenvalues of A_Λ counting multiplicities. It can be show that [10]

$$
\rho_\Lambda = \bigotimes_{i=1}^{\infty}((1-\gamma_i)P_{0,i} + \gamma_i P_{1,i})
$$

where $P_{0,i}$ and $P_{1,i}$ are projection operators in two dimensional space such that $P_{0,i} + P_{1,i} = 1$. The expression for the CCR algebra follows from the above representation. For the details we refer to Ref.[10].

Finally we show that the finiteness of S_Λ. We will use the following inequality [1]. For any trace class positive operator A and B the inequality

$$
-Tr(A\log A - A\log B) \leq Tr(B-A) \tag{3.12}
$$

holds.

We first show that

$$-Tr(A_\Lambda \log A_\Lambda) < \infty \tag{3.13}$$

Let $\Lambda = (-L/2, L/2)^\nu$ be the rectangular box, and let $\Delta_{\Lambda,P}$ be the Laplacian on Λ with periodic boundary condition. Choose positive constants α' and α such that $0 < \alpha' < \alpha < 1$, where α satisfies Assumption A(b). Denote

$$D_\Lambda \equiv \exp\{-(-\Delta_{\Lambda,P})^{\alpha'/2}\}$$

Then by (3.12) we have that

$$\begin{aligned}-Tr(A_\Lambda \log A_\Lambda) &= -Tr(A_\Lambda \log A_\Lambda - A_\Lambda \log D_\Lambda) - Tr(A_\Lambda \log D_\Lambda)\\ &\leq Tr(D_\Lambda - A_\Lambda) + Tr\big((-\Delta_{\Lambda,P})^{\alpha'/2} A_\Lambda\big)\end{aligned}$$

Since D_Λ and A_Λ are trace class operators, $Tr(D_\Lambda - A_\Lambda) < \infty$. Let

$$f_n(x) \equiv \prod_{j=1}^{\nu} \{L^{-1/2} \exp(i2n_j\pi x/L)\}, \quad n \in Z^\nu \tag{3.14}$$

Then $\{f_n\}$ form a basis for $L^2(\Lambda)$. Denote that

$$\widehat{\chi}_{\Lambda,n}(k) = \prod_{j=1}^{\nu} \left\{ \frac{2}{\sqrt{2\pi L}} \sin\big(\frac{L}{2}(k_j - \frac{2n_j\pi}{L})/(k_j - \frac{2n_j\pi}{L})\big) \right\} \tag{3.15}$$

A direct computation yields that

$$Tr\big((-\Delta_{\Lambda,P})^{\alpha'/2} A_\Lambda\big) = \sum_{n\in Z^\nu} \big(\frac{2|n|\pi}{L}\big)^{\alpha'} \int (\widehat{\chi}_{\Lambda,n}(k))^2 \widehat{K}(k) d^\nu k \tag{3.16}$$

where $|n|^2 = n_1^2 + \cdots + n_\nu^2$. Using Assumption A(b) and the fact that

$$|\widehat{\chi}_{\Lambda,n}(k)| \leq c \prod_{j=1}^{\nu} \big(1 + |k_j - 2n_j\pi/L|\big)^{-1}$$

it is easy to check that

$$\int \widehat{\chi}_{\Lambda,n}(k)^2 \widehat{K}(k) d^\nu k \le M \prod_{j=1}^{\nu} \left(1 + |2n_j\pi/L|\right)^{-(1+\alpha)} \tag{3.17}$$

for some $0 < M$. Substituting the above bound to (3.16) we prove that

$$Tr\left((-\Delta_{\Lambda,P})^{\alpha'/2} A_\Lambda\right) < \infty$$

This proved (3.13) completely

Next we note that

$$\begin{aligned} Tr((1+A_\Lambda)\log(1+A_\Lambda)) &\le Tr(A_\Lambda + A_\Lambda^2) \\ &\le (1 + \|A_\Lambda\|) Tr(A_\Lambda) \\ &< \infty \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} -Tr\left((1-A_\Lambda)\log(1-A_\Lambda)\right) &\le Tr(A_\Lambda) \\ &< \infty \end{aligned} \tag{3.19}$$

The finiteness of S_Λ follows from the first part of Proposition, (3.13) and (3.17)-(3.18). This completes the proof.

4. Estimate on Mean Entropy

In this section we prove Theorem 2.1. We first consider the mean entropy of the quasi-free state ω_A on the CCR algebra. As in Section 3, let $\Lambda = (-L/2, L/2)^\nu$ be a box. We introduce an operator E_Λ on $L^2(\Lambda)$ by

$$E_\Lambda \equiv \sum_{n \in Z^\nu} \widehat{K}(\frac{2n\pi}{L}) P_n \tag{4.1}$$

where for each $n \in Z^\nu$, P_n be the project operator onto one dimensional subspace spanned by the vector defined in (3.14). Put

$$\widetilde{S}_\Lambda \equiv -Tr(E_\Lambda \log E_\Lambda) + Tr\left((1+E_\Lambda)\log(1+E_\Lambda)\right). \tag{4.2}$$

We then have the following result:

PROPOSITION 4.1. *Under the Assumption A, the equality*

$$\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}S_\Lambda = \lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\widetilde{S}_\Lambda$$

holds for thr CCR algebra. An analogous result holds for CAR algebra.

Proof of Theorem 2.1. Notice that

$$-Tr(E_\Lambda \log E_\Lambda) = -\sum_{n\in Z^\nu}\widehat{K}(2n\pi/L)\log(\widehat{K}(2n\pi/L)).$$

REMARK. Since $-\widehat{K}\log\widehat{K} \le C(\delta)\widehat{K}^{1-\delta}$ for any $\delta > 0$. Assumption A(b) implies that $-\widehat{K}\log\widehat{K}$ is integrable. Thus it follows that

$$-\frac{1}{|\Lambda|}Tr(E_\Lambda \log E_\Lambda) \longrightarrow -\frac{1}{(2\pi)^\nu}\int \widehat{K}(k)\log\widehat{K}(k)d^\nu k$$

as $L \to \infty$. The second term in (4.2) gives the corresponding second term in the theorem. The method similar to that used in the above can be applied to the proof of the theorem for CAR. This proved Theorem 2.1 completely.

The rest of this paper is devoted to the proof of the Proposition 4.1. For each L, let $a(L)$ be a positive number satisfying

$$\lim_{L\to\infty} a(L) = 0. \tag{4.3}$$

We will specify the number $a(L)$ later. As in Section 3 we define an operator D_Λ on $L(\Lambda)$ by

$$D_\Lambda \equiv \sum_{n\in Z^\nu}\exp\{-|\frac{2n\pi}{L}|^{\alpha'}\}P_n \tag{4.4}$$

where $|a|$ is the Euclidian norm of $a \in R^\nu$, and $0 < \alpha' < \alpha < 1$. For a notational simplification, denote

$$F_\Lambda \equiv E_\Lambda + a(L)D_\Lambda \tag{4.5}$$

We then have that

$$
\begin{aligned}
-Tr(A_\Lambda \log A_\Lambda) &= -Tr(A_\Lambda \log A_\Lambda - A_\Lambda \log F_\Lambda) \\
&\quad - Tr\big((A_\Lambda - F_\Lambda)\log F_\Lambda\big) \\
&\quad - \big(Tr(F_\Lambda \log F_\Lambda) - Tr(E_\Lambda \log E_\Lambda)\big) \\
&\quad - Tr(E_\Lambda \log E_\Lambda) \\
&\equiv I_\Lambda + II_\Lambda + III_\Lambda + IV_\Lambda
\end{aligned} \tag{4.6}
$$

We assert that

$$
|\Lambda|^{-1}\big(|I_\Lambda| + |II_\Lambda| + |III_\Lambda\big) \to 0 \tag{4.7}
$$

as $L \to \infty$. Under the assertion we conclude that

$$
\big|\Lambda|^{-1}| - Tr(A_\Lambda \log A_\Lambda) + Tr(E_\Lambda \log E_\Lambda)\big| \to 0 \tag{4.8}
$$

as $L \to \infty$.

Let us prove our assertion in (4.7). Using (3.12) we obtain that

$$
\begin{aligned}
|I_\Lambda| &\le |Tr(F_\Lambda - A_\Lambda)| \\
&= |Tr(E_\Lambda + a(L)D_\Lambda - A_\Lambda)|
\end{aligned}
$$

Since $|\Lambda|^{-1}|Tr(E_\Lambda - A_\Lambda)| \to 0$ and $|\Lambda|^{-1}a(L)Tr(D_\Lambda \to 0$ as $L \to \infty$, $|\Lambda|^{-1}|I_\Lambda| \to 0$ as $L \to \infty$. Similaly it is easy to show that $|\Lambda|^{-1}|III_\Lambda| \to 0$ as $L \to \infty$.

Next we consider II_Λ. Since

$$
\big|(f_n, \log(E_\Lambda + a(L)D_\Lambda)f_n)\big| \le \big|(f_n, \log(a(L)D_\Lambda)f_n)\big|
$$

if $\big(f_n, (E_\Lambda + a(L)D_\Lambda)f_n\big) \le 1$, and $\big|(f_n, \log(E_\Lambda + a(L)D_\Lambda)f_n)\big|$ is bounded uniformly in Λ if $(f_n, (E_\Lambda + a(L)D_\Lambda)f_n) \ge 1$ by Assumption A(a), we have that

$$
|\Lambda|^{-1}|II_\Lambda| \le |\Lambda|^{-1} \sum_{n \in Z^\nu} \big|(f_n, (A_\Lambda - E_\Lambda)f_n)\big| [-\log a(L) + |\frac{2n\pi}{L}|^\alpha] \tag{4.9}
$$

Consider the quantity defined by

$$
\begin{aligned}
b(L) &\equiv |\Lambda|^{-1} \sum_{n\in Z^\nu} |\frac{2n\pi}{L}|^{\alpha'} \big|(f_n,(A_\Lambda - E_\Lambda)f_n)\big| \\
&= |\Lambda|^{-1} \sum_{n\in Z^\nu} |\frac{2n\pi}{L}|^{\alpha'} \Big| \int |\widehat{\chi}_{\Lambda,n}(k)|^2 \widehat{K}(k)d^\nu k - \widehat{K}(\frac{2n\pi}{L})\Big|
\end{aligned}
\tag{4.10}
$$

where $\widehat{\chi}_{\Lambda,n}$ has been defined in (3.15). Change of variable yield that

$$
\begin{aligned}
\widehat{K}_{L,n} &\equiv \int \big|\widehat{\chi}_{\Lambda,n}(k)\big|^2 \widehat{K}(k)d^\nu k \\
&= \int (\prod_{j=1}^{\nu} \frac{2}{\pi L} \sin^2(Lk_j/2)/k_j^2)\widehat{K}(k+(2n\pi/L))d^\nu k \\
&= \pi^{-\nu} \int (\prod_{j=1}^{\nu} \sin^2(k_j)/k_j^2)\widehat{K}((2k/L)+(2n\pi/L))d^\nu k
\end{aligned}
\tag{4.11}
$$

Note that $\pi^{-1}\int(\sin^2 k)k^{-2}dk = 1$. For given $k' \in R^\nu$, take $L \to \infty$ and $|n| \to \infty$ such that $\lim(2n\pi/L) = k'$. Then by the Dominated Convergence Theorem we obtain that

$$
\widehat{K}_{L,n} \longrightarrow \widehat{K}(k')
\tag{4.12}
$$

as $L \to \infty$ and $|n| \to \infty$. Here we have used Assumption A(b) and (c). Define

$$
\begin{aligned}
\widehat{K}_L^{(1)}(k') &= |\frac{2n\pi}{L}|^{\alpha'} \widehat{K}_{L,n} && \text{if} \quad k' \in [\frac{2n\pi}{L}, \frac{2(n+1)\pi}{L}) \\
\widehat{K}_L^{(2)}(k') &= |\frac{2n\pi}{L}|^{\alpha'} \widehat{K}(2n\pi/L) && \text{if} \quad k' \in [\frac{2n\pi}{L}, \frac{2(n+1)\pi}{L})
\end{aligned}
$$

Then

$$
b(L) = \frac{1}{(2\pi)^\nu} \int |\widehat{K}_L^{(1)}(k') - \widehat{K}_L^{(2)}(k')| d^\nu k'
$$

If $0 < \alpha' < \alpha < 1$, one may use Assumption A(b), (3.17), the Dominated Convergence Theorem and (4.12) to conclude that

$$
b(L) \longrightarrow 0 \quad \text{as} \quad L \to \infty
\tag{4.13}
$$

Similarly it can be check that (the case in which $\alpha' = 0$ in (4.12))

$$c(L) \equiv |\Lambda|^{-1} \sum_{n \in Z^{\nu}} |(f_n, (A_\Lambda - E_\Lambda) f_n)| \longrightarrow 0 \tag{4.14}$$

as $L \to \infty$. We choose $a(L)$ such that

$$c(L)(-\log a(L)) \to 0 \tag{4.15}$$

Then it follow from (4.9), (4.10), (4.13), (4.14) and (4.15) that

$$|\Lambda|^{-1} |II_\Lambda| \to 0$$

as $L \to \infty$. This proved the assertion in (4.7).

It is clear that a straight forward application of the method used in the proof of (4.8) show that

$$|Tr((1 \pm A_\Lambda) \log(1 \pm A_\Lambda) - Tr((1 \pm E_\Lambda \log(1 \pm E_\Lambda))| \to 0$$

as $L \to \infty$. We leave the detailed proof of the above result to the reader. This completes the proof of the Proposition.

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