Comm. Korean Math. Soc. 6(1991), No. 2, pp. 267–281

A G/M/1 VACATION MODEL WITH EXHAUSTIVE SERVER

BONG DAE CHOI AND KWANG KYU PARK

1. Introduction

In recent years there have been significant contributions to the theory of queue with server vacations. For complete reference on vacation model, see Doshi [6]. Vacation models have been widely used to model many problems in computer, communication and production systems. Vacation models also are closely related to cyclic queues, priority queues and retrial queues. Different vacation models are distinguished by their scheduling disciplines, i.e., by the rules which determine when service stops and a vacation begins. In the exhaustive service discipline, the server takes a vacation if the system is empty. In a limited service discipline, the server takes a vacation when the system is empty or when K customers have been served during the current visit, whichever occurs first. In the Bernoulli schedule discipline, the server begins a vacation either if the system is empty or if, at a service completion, the system is not empty, then sevice is resumed with fixed probability p and the server takes vacation with probability $1-p$. $M/G/1$ vacation model under the exhaustive service discipline has been investigated by Levy and Yechialli [13], Heyman [9], Lee [12], Doshi [5,6], Fuhrmann and Cooper [7]. $M/G/1$ vacation model under the limited service discipline has been studied by Cramer [3] and others. $M/G/1$ vacation model under the Bernoulli schedule discipline has been studied by Keilson and Servi [10], Ramaswamy and Servi [16].

Recently J.K. Daniel and Krishnamoorthy [4] investigated $G/M/1$ vacation model with a limited service by the matrix-geometric approach.

Received July 4, 1991.

Supported in part by a grant from Korea Science and Engineering Foundation, 1990-1993.

In this paper we consider a $G/M/1$ queue with a exponential vacation time and an exhaustive service. Even under the exhaustive service disciplines there are two different vacation models which are called the multiple vacation model and the single vacation model. In the multiple vacation model the server takes a vaction every time the system becomes empty and whenever the server returns from a vacation to find no waiting customers. In the single vacation model the server takes single vacation only when the system becomes idle at the end of busy period. If, on return from a vacation, the system is empty, then the server waits his return to serve customers.

We analyse the multiple vacation model for $G/M/1$ queue by imbedded Markov chain approach in the section 2. Queue length probabilities at arrival time points are derived explicitly by solving a nonhomogeneous difference equation. Queue length probabilities at arbitrary time points are obtained in the section 3. The formula for the Laplace transform of waiting time shows that the stochastic decomposition property holds. The single vacation model for the $G/M/1$ queue is analysed briefly in the section 5.

2. Queue length probabilities at arrival points

First let us describe the multiple vacation model in detail. When the server finishes serving a customer and finds the system empty, he goes away for a random length of time called vacation. The vacation time is utilized for secondary customers or maintenence jobs. If the server returns from the vacation and finds at least one customer waiting, he works until the system empties, then he takes another vacation. If the server returns from the vacation and finds no customer waiting, he immediately takes another vacation, and continues in this manner until he finds at least one customer waiting, when he returns from vacation.

We consider a $G/M/1$ queueing system with the server vacations in which customer arrives according to a renewal process with interarrival time distribution $A(t)$, service times are distributed exponentially with mean $\frac{1}{\mu}$ and vacation times are distributed exponentially with mean $\frac{1}{\alpha}$. Let $A^*(s) = \int_0^\infty e^{-sx} dA(x)$ be the Laplace transform of $A(t)$ and $\frac{1}{\lambda}$ the mean of $A(t)$.

Observing the vacation system immediately prior to an arrival time point we obtain an imbedded Markov chain defined on the state space $S = \{(i,j)|i = 0,1, j = 0,1,2,...\} - \{(1,0)\}$. $i = 0$ indicates that the server is on vacation, and $i = 1$ indicates the availability of the server at the service facility. The index j indicates the number of customers immediately prior to an arrival. Note that state (1,0) is not appeared in the multiple vacation model, because it is impossible that the server is available and the system is empty at the moment of an arrival. The transition probability matrix P associated with the imbedded Markov chain is given by

$$P = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,1) \\ (0,2) \\ (1,2) \\ (0,3) \\ (1,3) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,0) & (0,1) & (1,1) & (0,2) & (1,2) & (0,3) & (1,3) & \cdots \end{array} \\ \begin{pmatrix} & A^*(\alpha) & a_0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & a_1 & A^*(\alpha) & a_0 & 0 & 0 & \cdots \\ & 0 & b_1 & 0 & b_0 & 0 & 0 & \cdots \\ & 0 & a_2 & 0 & a_1 & A^*(\alpha) & a_0 & \cdots \\ & 0 & b_2 & 0 & b_1 & 0 & b_0 & \cdots \\ & 0 & a_3 & 0 & a_2 & 0 & a_1 & \cdots \\ & 0 & b_3 & 0 & b_2 & 0 & b_1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

The element $A^*(\alpha)$ is the probability that the vacation is continued in an interarrival time,

$$A^*(\alpha) = \int_0^\infty e^{-\alpha x}\, dA(x).$$

The element a_k is the probability that the vacation is over and k customers are served during an interarrival time. If we assume that the interarrival time is x and it requires $x - t$ sec. until the vacation is terminated, then a_k is given by

$$a_k = \int_0^\infty \int_0^x \alpha e^{-\alpha(x-t)} \frac{e^{-\mu t}(\mu t)^k}{k!}\, dt\, dA(x), k = 0,1,2,...$$

The element b_k is the probability that k customers are served during an interarrival time,

$$b_k = \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!}\, dA(x), k = 0,1,2,...$$

The elements in the first column are the numbers which make each row probability distribution.

It is well known (see Prabhu [14]) that the imbedded Markov chain is ergodic if and only if $\rho = \frac{\lambda}{\mu} < 1$. In the rest of this paper we always assume that $\rho < 1$.

In steady state let N be the number of customers present immediately prior to arrival time points and i the status of the server. Define

$$\pi_n = P(i = 0, N = n),$$
$$\omega_n = P(i = 1, N = n),$$

i.e. π_n[resp. ω_n] is the steady-state probability that an arrival sees n customers in the system and the server is on vacation [resp. available]. Then $\{\pi_n\}$ and $\{\omega_n\}$ satisfy the stationary equation;

$$(1) \qquad (\pi_0, \pi_1, \omega_1, \pi_2, \omega_2, \cdots)P = (\pi_0, \pi_1, \omega_1, \pi_2, \omega_2, \cdots),$$

and

$$\sum_{i=0}^{\infty} \pi_i + \sum_{i=1}^{\infty} \omega_i = 1.$$

Now the equation (1) may be written as

$$(2a) \qquad \pi_n = A^*(\alpha)\pi_{n-1}, n = 1, 2, \cdots ;$$

$$(2b) \qquad \omega_n = \sum_{k=0}^{\infty} b_k \omega_{n+k-1} + \sum a_k \pi_{n+k-1}, \quad n = 1, 2, \cdots$$

From (2a) we obtain

$$(3a) \qquad \pi_n = \pi_0 (A^*(\alpha))^n, \quad n \geq 0.$$

Note that

$$\sum_{k=0}^{\infty} a_k \pi_{n+k-1}$$
$$= \sum_{k=0}^{\infty} \pi_0 (A^*(\alpha))^{n+k-1} \int_0^{\infty} \int_0^{x} \alpha e^{-\alpha(x-t)} \frac{e^{-\mu t}(\mu t)^k}{k!} \, dt \, dA(x)$$
$$= \pi_0 A^*(\alpha)^{n-1} \int_0^{\infty} \alpha e^{-\alpha x} \int_0^{x} e^{-(\mu-\alpha-\mu A^*(\alpha))t} \, dt \, dA(x)$$
$$= \begin{cases} \pi_0 A^*(\alpha)^{n-1} \alpha \frac{A^*(\alpha)-A^*(\mu-\mu A^*(\alpha))}{\mu-\alpha-\mu A^*(\alpha)}, & \text{if } \mu-\alpha-\mu A^*(\alpha) \neq 0, \\ \pi_0 A^*(\alpha)^{n-1} \alpha \int_0^{\infty} x e^{-\alpha x} \, dA(x), & \text{if } \mu-\alpha-\mu A^*(\alpha) = 0 \,. \end{cases}$$

We have two cases depending on whether $\mu-\alpha-\mu A^*(\alpha)$ is zero or not. But we think of $\frac{A^*(\alpha)-A^*(\mu-\mu A^*(\alpha))}{\mu-\alpha-\mu A^*(\alpha)}$ as function of μ, then

$$\lim \frac{A^*(\alpha)-A^*(\mu-\mu A^*(\alpha))}{\mu-\alpha-\mu A^*(\alpha)} = \int_0^{\infty} x e^{\alpha x} \, dA(x)$$

when $\mu-\alpha-\mu A^*(\alpha)=0$. Thus the rsults for the case $\mu-\alpha-\mu A^*(\alpha)=0$ follows from these for the case $\mu-\alpha-\mu A^*(\alpha) \neq 0$ by using L'Hospital rule. So only the case $\mu-\alpha-\mu A^*(\alpha) \neq 0$ is treated in detail.

Now we have from (2b) for $n \geq 1$

$$\text{(3b)} \qquad \omega_n - \sum_{l=0}^{\infty} \omega_{n+l-1} b_l = \pi_0 A^*(\alpha)^{n-1} \alpha \frac{A^*(\alpha)-A^*(\mu-\mu A^*(\alpha))}{\mu-\alpha-\mu A^*(\alpha)}.$$

Next we employ the method of the shift operator to obtain solution $\{\omega_n\}$ of (3b). Letting $D\omega_n = \omega_{n+1}$, we find that (3b) can be written as

$$\text{(4)} \qquad (D - \sum_{l=0}^{\infty} b_l D^l)\omega_{n-1} = \pi_0 A^*(\alpha)^{n-1} \alpha \frac{A^*(\alpha)-A^*(\mu-\mu A^*(\alpha))}{\mu-\alpha-\mu A^*(\alpha)}.$$

So the limiting probabilities $\{\omega_n\}$ are the solution of the non-homogeneous difference equation (4) subject to the boundary condition (3b) for $n=1$. First we need to find the general solution of (4) which is the sum of the general solution to the homogeneous equation (right hand side of (4) replaced by zero) and a particular solution to (4). It is well known (Gross and Harris [8, p.307]) that the general solution of homogeneous equation $(D - \sum_{l=0}^{\infty} b_l D^l)\omega_{n-1} = 0$ is given by $\omega_n = c\gamma^n$, where γ is the unique solution in the unit disc of the equation $z - \sum_{l=0}^{\infty} b_l z^l = 0$, $0 < \gamma < 1$, whose existence and uniqueness is guarented by the traffic condition $\rho = \frac{\lambda}{\mu} < 1$. Usually it is not easy to find a particular solution of the difference equation. One particular solution of (4) will be given of the form

$$\omega_n = d\sum_{l=0}^{n-1} \gamma^{n-1-l} A^*(\alpha)^l = d\frac{\gamma^n - A^*(\alpha)^n}{\gamma - A^*(\alpha)}, \tag{5}$$

where d is a constant to be determined by substituting (5) into (3b). In fact , guess to this solution (5) comes from the operational calculus.

We calculate the left hand side of (3b) to find out the constant d.

$$\begin{aligned}
&\omega_n - \sum_{l=0}^{\infty} \omega_{n+l-1} b_l \\
&= d\frac{\gamma^n - A^*(\alpha)^n}{\gamma - A^*(\alpha)} - d\sum_{l=0}^{\infty} \frac{\gamma^{n+l-1} - A^*(\alpha)^{n+l-1}}{\gamma - A^*(\alpha)} b_l \\
&= d\frac{\gamma^n - A^*(\alpha)^n}{\gamma - A^*(\alpha)} - \frac{d\gamma^{n-1}}{\gamma - A^*(\alpha)} \sum_{l=0}^{\infty} \gamma^l b_l + \frac{dA^*(\alpha)^{n-1}}{\gamma - A^*(\alpha)} \sum_{l=0}^{\infty} A^*(\alpha)^l b_l \\
&= d\frac{\gamma^n - A^*(\alpha)^n}{\gamma - A^*(\alpha)} - \frac{d\gamma^n}{\gamma - A^*(\alpha)} + dA^*(\alpha)^{n-1} \frac{A^*(\mu - \mu A^*(\alpha))}{\gamma - A^*(\alpha)} \\
&= d\frac{A^*(\mu - \mu A^*(\alpha)) - A^*(\alpha)}{\gamma - A^*(\alpha)} A^*(\alpha)^{n-1}.
\end{aligned}$$

Thus we obtain

$$d = \begin{cases} \frac{A^*(\alpha) - \gamma}{\mu - \alpha - \mu A^*(\alpha)} \pi_0 \alpha & \text{if } \mu - \alpha - \mu A^*(\alpha) \neq 0, \\ \frac{\int x e^{\alpha x}\, dA(x)}{1 - \mu \int_0^{\infty} x e^{-\alpha x}\, dA(x)} \pi_0 \alpha & \text{if } \mu - \alpha - \mu A^*(\alpha) = 0. \end{cases}$$

Hence the general solution of (4) is given by

$$\omega_n = c\gamma^n + d\sum_{l=0}^{n-1} \gamma^{n-1-l} A^*(\alpha)^l, \qquad n \geq 1. \tag{7}$$

The constant c is determined so that (7) satisfies the boundary condition (3b) with $n = 1$. By substituting (7) into (3b) with $n = 1$, the left hand side of (3b) is equal to

$$\begin{aligned}
&\omega_1 - \sum_{l=0}^{\infty} \omega_l b_l \\
&= c\gamma + d - \sum_{l=1}^{\infty} (c\gamma^l + d\frac{\gamma^l - A^*(\alpha)^l}{\gamma - A^*(\alpha)}) b_l \\
&= c\gamma + d - c\gamma + cA^*(\mu) - d\left(\frac{\gamma}{\gamma - A^*(\alpha)} - \frac{A^*(\mu - \mu A^*(\alpha))}{\mu - A^*(\alpha)}\right) \\
&= cA^*(\mu) + \frac{d}{\gamma - A^*(\alpha)}(A^*(\mu - \mu A^*(\alpha)) - A^*(\alpha)) \\
&= cA^*(\mu) + \frac{A^*(\alpha) - A^*(\mu - \mu A^*(\alpha))}{\mu - \alpha - \mu A^*(\alpha)} \pi_0 \alpha
\end{aligned}$$

Thus we have from (3b) with $n = 1$ that $cA^*(\mu) = 0$, and hence $c = 0$. It remains to find out π_0. From $\sum_{n=0}^{\infty} \pi_n + \sum_{n=1}^{\infty} \omega_n = 1$, we obtain with some calculation that

$$\pi_0 = \begin{cases} \frac{(\mu - \alpha - \mu A^*(\alpha))(1-\gamma)}{\mu - \alpha - \gamma\mu} & \text{if } \mu - \alpha - \mu A^*(\alpha) \neq 0; \\ 1 - \gamma & \text{if } \mu - \alpha - \mu A^*(\alpha) = 0. \end{cases} \tag{8}$$

We need the following remark to make sure that constant d given by (6) and the probability π_0 are positive.

Note that $\sum_{l=0}^{\infty} b_l z^l = A^*(\mu - \mu z)$ and so γ is the solution of $z - A^*(\mu - \mu z) = 0$. If $\mu - \alpha - \mu A^*(\alpha) = 0$ then $\alpha = \mu - \mu A^*(\alpha)$ and so $A^*(\alpha) = A^*(\mu - \mu A^*(\alpha))$. By uniqueness of the solution, we have $A^*(\alpha) = \gamma$. If $\mu - \alpha - \mu A^*(\alpha) > 0$ then $A^*(\alpha) > A^*(\mu - \mu A^*(\alpha))$, and so $\gamma < A^*(\alpha)$, and so $0 < \mu - \alpha - \mu A^*(\alpha) < \mu - \alpha - \mu\gamma$. This confirms that d and π_0 are positive.

Thus we have the following:

THEOREM 1. *Assume that $\frac{\lambda}{\mu} < 1$ and let γ be the unique solution of $z - A^*(\mu - \mu z) = 0$. Let π_n[resp. ω_n] be the steady-state probability that an arrival sees n customers in the system and the server is on the vacation [resp. available]. Then*

$$\pi_n = \pi_0 A^*(\alpha)^n,$$
$$\omega_n = d \sum_{k=0}^{n-1} \gamma^{n-1-k} A^*(\alpha)^k,$$

where π_0 is given by (8) and

$$d = \begin{cases} \frac{A^*(\alpha)-\gamma}{\mu-\alpha-\gamma\mu}(1-\gamma)\alpha & \text{if } \mu - \alpha - \mu A^*(\alpha) \neq 0, \\ \frac{\int_0^\infty x e^{\alpha x}\, dA(x)}{1-\mu \int_0^\infty x e^{-\mu x}\, dA(x)} & \text{if } \mu - \alpha - \mu A^*(\alpha) = 0. \end{cases}$$

If $\alpha \to \infty$, then our vacation model is reduced to the ordinary $G/M/1$ queue without vacation in the sense that queue size distribution for $G/M/1$ queue with vacation approaches to that for $G/M/1$ queue without vacation. The following corollary shows the above fact holds.

COROLLARY 2. *If $\alpha \to \infty$ in the above vacation model, then*

$$\pi_0 \to 1 - \gamma,$$
$$\pi_n \to 0, \quad n \geq 1,$$
$$\omega_n \to (1-\gamma)\gamma^n, \quad n \geq 1.$$

Proof. For sufficient large $\alpha, \mu - \alpha - \mu A^*(\alpha) \neq 0$. Hence

$$\lim_{\alpha \to \infty} \pi_0 = \lim_{\alpha \to \infty} \frac{(\mu - \alpha - \mu A^*(\alpha))(1-\gamma)}{\mu - \alpha - \gamma\mu} = 1 - \gamma,$$
$$\lim_{\alpha \to \infty} d = \lim_{\alpha \to \infty} \frac{A^*(\alpha) - \gamma}{\mu - \alpha - \mu A^*(\alpha)} \pi_0 \alpha = (1-\gamma)\gamma.$$

Thus the required result is obtained.

3. Queue length probabilities at arbitrary time points

Let $U(t)$ be the distribution of the equilibrium backward recurrence time of an interarrival time; that is,

$$U(t) = \lambda \int_0^t (1 - A(x))\, dx.$$

Let p_k [resp. q_k] be the equibrium probability that at arbitrary time point there are k customers in the system and the server is on vacation [resp. available]. Then these probabilities are given by (Gross and Harris [8, p.152])

$$\begin{aligned} &p_n(\text{ or } q_n\ ) \\ &= \sum_{i=0}^{\infty} \left[\pi_i \int_0^\infty P\{\text{appropriate changes in } t \text{ to bring state from } i \text{ to } n\}\, dU(t) \right] \\ &+ \sum_{i=0}^{\infty} \left[\omega_i \int_0^\infty P\{\text{appropriate changes in } t \text{ to bring state from } i \text{ to } n\}\, dU(t) \right]. \end{aligned}$$

In the process of calculating p_n and q_n, we often use the formula

$$\int_0^\infty e^{-\theta x}(1 - A(x))\, dx = \frac{1 - A^*(\theta)}{\theta}.$$

Let us find out $p_n (n \geq 1)$.

$$\begin{aligned} p_n &= \pi_{n-1} \int_0^\infty P(\text{the vacation time} > t)\, dU(t) \\ &= \lambda \pi_{n-1} \int_0^\infty e^{-\alpha t}(1 - A(t))\, dt \\ &= \lambda \pi_{n-1} \frac{1 - A^*(\alpha)}{\alpha} \\ &= \lambda \pi_0 A^*(\alpha)^{n-1} \frac{1 - A^*(\alpha)}{\alpha}. \end{aligned}$$

Next we calculate q_n $(n \geq 1)$.

$$\begin{aligned}
q_n &= \sum_{k=0}^{\infty} \omega_{n+k-1} \int_0^\infty P\{k \text{ departures in } [0,t]\}\, dU(t) \\
&\quad + \sum_{k=0}^{\infty} \pi_{n+k-1} \int_0^\infty P\{\text{vacation is over and } k \text{ departures in } [0,t]\}\, dU(t) \\
&= \lambda \sum_{k=0}^{\infty} \omega_{n+k-1} \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!}(1-A(t))\,dt \\
&\quad + \sum_{k=0}^{\infty} \pi_{n+k-1} \int_0^\infty \int_0^t \alpha e^{-\alpha s} \frac{e^{-\mu(t-s)}(\mu(t-s))^k}{k!}\, ds (1-A(t))\,dt \\
&= \lambda d \sum_{k=0}^{\infty} \frac{\gamma^{n+k-1} - A^*(\alpha)^{n+k-1}}{\gamma - A^*(\alpha)} \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!}(1-A(t))\,dt \\
&\quad + \lambda \pi_0 \sum_{k=0}^{\infty} A^*(\alpha)^{n+k-1} \int_0^\infty \int_0^t \alpha e^{\alpha s} \frac{e^{-\mu(t-s)}(\mu(t-x))^k}{k!}\, ds (1-A(t))\,dt \\
&= \lambda d \frac{1}{\gamma - A^*(\alpha)} \left\{ \gamma^{n-1} \int_0^\infty e^{-(\mu-\mu\gamma)t}(1-A(t))\,dt \right. \\
&\quad \left. - A^*(\alpha)^{n-1} \int_0^\infty e^{-(\mu-\mu A^*(\alpha))t}(1-A(t))\,dt \right\} \\
&\quad + \lambda \pi_0 A^*(\alpha)^{n-1} \int_0^\infty \frac{\alpha e^{\alpha t} - \alpha e^{-(\mu-\mu A^*(\alpha))t}}{\mu - \alpha - \mu A^*(\alpha)}(1-A(t))\,dt \\
&\quad + \frac{\lambda d}{\gamma - A^*(\alpha)} \left\{ \gamma^{n-1} \frac{1-A^*(\mu-\mu\gamma)}{\mu - \mu\gamma} - A^*(\alpha)^{n-1} \frac{1 - A^*(\mu - \mu A^*(\alpha))}{\mu - \mu A^*(\alpha)} \right\} \\
&\quad + \lambda \pi_0 A^*(\alpha)^{n-1} \frac{\alpha}{\mu - \mu A^*(\alpha)} \left\{ \frac{1-A^*(\alpha)}{\alpha} - \frac{1-A^*(\mu-\mu A^*(\alpha))}{\mu - \mu A^*(\alpha)} \right\}.
\end{aligned}$$

After substituting d and π_0 and lengthy calculation, we obtain

$$q_n = \frac{\lambda(1-\gamma)\alpha}{\mu - \alpha - \mu\gamma} \left(\frac{\gamma^{n-1}}{\mu} - \frac{1-A^*(\alpha)}{\alpha} A^*(\alpha)^{n-1} \right).$$

Returning now to p_o, we have

$$p_0 = \sum_{k=1}^{\infty}\left\{\omega_k \int_0^\infty P\{\text{at least } k+1 \text{ departures in } [0,t]\}\, dU(t)\right\}$$
$$+\sum_{k=0}^{\infty}\left\{\pi_k \int_0^\infty P\{\text{vacation is over and at least } k+1 \text{ departure}\}\, dU(t)\right\}$$
$$= \lambda \sum_{k=1}^{\infty} \frac{\gamma^k - A^*(\alpha)^k}{\gamma - A^*(\alpha)} \int_0^\infty \int_0^t \frac{\mu e^{-\mu s}(\mu x)^t}{k!}\, ds(1-A(t))\, dt$$
$$+ \lambda\pi_0 \sum_{k=0}^{\infty} A^*(\alpha)^k \int_0^\infty \int_0^t \alpha e^{-as} \int_0^{t-x} \frac{\mu e^{-\mu s}(\mu x)^k}{k!}\, dx\, ds(1-A(t))dt.$$

With tedious calculation, we obtain

$$p_0 = \frac{\alpha}{\alpha+\gamma\mu-\mu}(1-\frac{\lambda}{\mu})$$
(9) $$+\frac{(1-\gamma)}{\mu-\alpha-\gamma\mu}\{\mu - \lambda(\mu-\alpha-\mu A^*(\alpha)) - \lambda\}.$$

Thus we have the following

THEOREM 3. *Let p_n[resp. q_n] be the equilibrium probability that at arbitrary time point there are k customers in the system and the server is on vacation [resp. busy period]. Then*

$$p_n = \frac{\lambda(\mu-\alpha-\mu A^*(\alpha))(1-\gamma)}{\mu-\alpha-\gamma\mu}\frac{1-A^*(\alpha)}{\alpha},$$
$$q_n = \frac{\lambda(1-\gamma)\alpha}{\mu-\alpha-\mu\gamma}\left(\frac{\gamma^{n-1}}{\mu} - \frac{1-A^*(\alpha)}{\alpha}A^*(\alpha)^{n-1}\right),$$

p_0 is given by (9).

REMARK. If $\alpha \to \infty$, then $q_n \to \frac{\lambda}{\mu}(1-\gamma)\gamma^{n-1}$, $p_0 \to 1-\frac{\lambda}{\mu}$ and $p_n \to 0 (n \geq 1)$. We see that the limiting queue size probability for vacation model as $\alpha \to \infty$ is the exact probability of number of customers at random time points in $G/M/1$ queue without vacation.

4. Waiting time distribution

Let W be the waiting time of an arriving customer in the queue. Then the Laplace transform of W can written as

$$\begin{aligned}
&E(e^{-\theta W}) \\
&= \sum_{k=0}^{\infty} E(e^{-\theta W}|\text{arrival finds } k \text{ customers and the server is on vacation})\pi_k \\
&+ \sum_{k=1}^{\infty} E(e^{-\theta W}|\text{arrival finds } k \text{ customers and the server is busy})\omega_k \\
&= \sum_{k=0}^{\infty}(\frac{\mu}{\mu+\theta})^k \frac{\alpha}{\alpha+\theta}\pi_k + \sum_{k=1}^{\infty}(\frac{\mu}{\mu+\theta})^k \omega_k.
\end{aligned}$$

Simple calculation yields

$$E(e^{\theta W}) = \frac{\alpha}{\alpha+\theta}\frac{1-\gamma}{1-\gamma\frac{\mu}{\mu+\theta}}.$$

The first factor on the right side of (10) is the Laplace transfrom of the remaining vacation time. The second factor on the right side of (10) is the Laplace transfrom of waiting time in the usual $G/M/1$ queue without vacation. This confirms that the stochastic decomposition property (Doshi [5]) holds for $G/M/1$ vacation model, i.e., the steady-state waiting time is the sum of two independent random variables; one of these is the waiting time in the same queue without vacation, and the other is the remaining vacation time.

5. G/M/1 with a single vacation

We now describe the single vacation model. The server takes exactly one vacation after the end of each busy peroid. If, on return from vacation, the server finds at least one customers waiting in queue, then he starts service immediately and keeps busy until the system becomes idle again and leaves for another vacation. If no customers have arrived during the vacation time, then the server waits for the next arrival as in the usual $G/M/1$ queue.

In the sequel we use the same notation as for $G/M/1$ with multiple vacation model. The state space is the set $\{(i,j)|i=0,1,j=0,1,2,3,\cdots\}$. The state stands for the same meaning as for multiple vacation model. Here the state (1,0) corresponds to the event that a customer arrives when the server is waiting for customer after a vaction.

The transition probability matrix P for the Markov chain obtained by observing the system immediatedly prior to an arrival time pont is given by

$$P = \begin{array}{c} \\ (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \\ (0,2) \\ (1,2) \\ (0,3) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,0) & (1,0) & (0,1) & (1,1) & (0,2) & (1,2) & (0,3) & \cdots \end{array} \\ \left(\begin{array}{cccccccc} & c_0 & A^*(\alpha) & a_0 & 0 & 0 & 0 & \cdots \\ & d_0 & 0 & b_0 & 0 & 0 & 0 & \cdots \\ & c_1 & 0 & a_1 & A^*(\alpha) & a_0 & 0 & \cdots \\ & d_1 & 0 & b_1 & 0 & b_0 & 0 & \cdots \\ & c_2 & 0 & a_2 & 0 & a_1 & A^*(\alpha) & \cdots \\ & d_2 & 0 & b_2 & 0 & b_1 & 0 & \cdots \\ & c_3 & 0 & a_3 & 0 & a_2 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \end{array}$$

The a_k, b_k are the same ones as in multiple vacation model. The element $A^*(\mu)$ is the probability that during an interarrival time there is no service completion,

$$A^*(\mu) = \int_0^\infty e^{-\mu t}\, dA(t).$$

The element c_k is the probability that $k+1$ customers are served and two vacations are over during an interarrival time,

$$c_k = \int_0^\infty \int_0^x \left(\int_0^{x-t} \frac{\mu e^{-\mu s}(\mu s)^k}{k!}\, ds \right) \alpha e^{-\alpha t}(\alpha t)\, dt\, dA(x).$$

The element d_k is the probability that $k+1$ customers are served and one vacatin is over during an interarrival time,

$$d_k = \int_0^\infty \int_0^x \left(\int_0^{x-t} \frac{\mu e^{\mu x}(\mu s)^k}{k!}\, ds \right) \alpha e^{-\alpha t}(\alpha t)\, dt\, dA(x).$$

Let $\{\pi_n\}$ and $\{\omega_n\}$ have the same meaning as in multiple vacation model. Then they satisfy the following:

$$\omega_o = \sum_{k=0}^{\infty}(\pi_k + d_k\omega_k) \tag{11.a}$$

$$\omega_1 = \omega_0 A^*(\mu) + \sum_{i=0}^{\infty} a_i\pi_i + \sum_{i=0}^{\infty} b_i\omega_i \tag{11.b}$$

$$\omega_n = \sum_{k=0}^{\infty} b_k\omega_{n+k-1} + \sum_{k=0}^{\infty} a_k\pi_{n+k-1} \quad (n \geq 2) \tag{11.c}$$

$$\pi_n = \pi_{n-1}A^*(\alpha) \quad (n \geq 1) \tag{11.d}$$

$$1 = \sum_{k=0}^{\infty}(\pi_k + \omega_k). \tag{11.e}$$

By the same way as in multiple vacation model, we have

$$\pi_n = \pi_o A^*(\alpha)^n (n \geq 0) \tag{12.a}$$

$$\omega_n = \alpha\pi_0 \frac{A^*(\alpha) - \gamma}{\mu - \alpha - \mu A^*(\alpha)} \sum_{k=0}^{n-l-1} \gamma^{n-k-1} A^*(\alpha)^k \quad (n \geq 2). \tag{12.b}$$

Now it remains to determine ω_0, ω_1 and π_0. But we have three equations (11.a), (11.b) and (11.c) and three unknown numbers. So we can obtain ω_0, ω_1 and π_0 as known parameters.

References

1. Chaudry, M. C. and Templeton, J. T. C. (1983), *A First Course in Bulk Queues*, Jonh Wiley and Sons, New York.
2. Cooper, R. B. (1981), *Introduction to Queueing Theory*, North-Holland.
3. Cramer, M. (1989), *Stationary distributions in a queueing syustem with vacation times and limited service*, Queueing Systems **4**, 457–68.

4. Daniel, J. T. and Krishnamoorhty, A. (1986), *$G/M/1$ queue with rest periods*, Optimization **17**, 535–543.
5. Doshi, B. T. (1985), *A note on stochastic decomposition in a $G/G/1$ queue with vacation or set up times*, J. Appl. Prob. **22**, 419–428.
6. ——— (1986), *Queueing systems with vacations – a survery*, Queueing Systems **1**, 29-66.
7. Fuhrmann, S. W. and Cooper, R. B. (1985), *Stochastic decompositiom in an $M/G/1$ queue with generalized vacations*, Oper. Res. **33**, 1117-1129.
8. Gross, D. and Harris (1985), *Fundamentals of Queueing Theroy*, John Wiley and Sons.
9. Heyman, D. P. (1980), *Optimal operating policies for $M/G/1$ queueing systems*, Oper. Res. **16**, 362-382.
10. Keilson, J. and Servi, L. D. (1986), *Oscillating random walk models for $GI/G/1$ vacation systems with Bernoulli schedules*, J. Appl. Prob. **23**, 790-802.
11. Krabbe, G. (1970), *Operational Calculus*, Springer–Verlag.
12. Lee, T. T. (1984), *$M/G/1/N$ queue with vacation time and exhaustive service discipline*, Oper. Res. **32**, 774-784.
13. Levy, Y. and Yechialli (1975), *Utilization of idle time in an $M/G/1$ queueing*, Mgt. Sci. **22**, 202-211.
14. Prabhu, N. H. (1980), *Stochastic storage Processes*, Springer–Verlag.
15. Ramaswamy, R. and Servi, L. D. (1988), *The busy period of the $M/G/1$ vacation model with a Bernoulli schedules*, Comm Statist Stochstic Model **4**, 507-521.

Department of Mathematics
Korea Advanced Institute of Science and Technology
P. O. Box 150, Seoul 130–650, Korea
and
Electronics and Telecommunications Research Institute
P. O. Box 8, Daeduk Science Town
Daejeon 305–606, Korea