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A DUALITY–LIKE OPERATION ON THE UNIPOTENT CHARACTERS OF A REDUCTIVE GROUP OVER A FINITE FIELD

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1. Introduction

Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$ with the connected center and a Frobenius morphism F and let G^F be the finite subgroup of F-fixed elements in G. Considerable progress in the representation theory of G^F has been made by G. Lusztig. He has shown that the knowledge of irreducible unipotent characters (see Section 3) of G^F is of primary importance in understanding the character ring of G^F.

Lusztig's Jordan decomposition of irreducible characters ([7]) implies, in particular, that there is a one-to-one correspondence between the set of irreducible unipotent characters of G^F and that of the dual group $(G^*)^{F^*}$. Let us call this bijection the *Lusztig's correspondence* in this paper.

The purpose of this paper is to give a simple alternative description of the Lusztig's correspondence under the assumption that all the irreducible unipotent characters of G^F are uniform characters (see Corollary 3.5). We also introduce a *duality-like* operation on the irreducible unipotent characters which coincides with the Lusztig's correspondence and commutes with the duality operation of Curtis[3], Alvis[1] and Kawanaka[6] (see Theorem 2.3 and Proposition 3.4).

We use the standard notation in [4] and [2]. All the characters are complex characters. For a finite group H, let 1_H (or 1) denote the principal character of H. For class functions φ and ψ of H, we define the ordinary hermitian inner product (φ, ψ) by $\frac{1}{|H|}\sum_{h\in H}\varphi(h)\overline{\psi(h)}$.

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2. Preliminaries

One of the most important ideas in the representation theory of G^F is the concept of dual groups. This was first appeared in Deligne-Lusztig[4]. Here we use the notation in [2, Ch. 4]. Let (G, F) and (G^*, F^*) be in duality with respect to maximally split torus T_0 and T_0^* respectively (see [2, p. 114]). Then this duality gives rise to a bijection between the G^F-conjugacy classes of F-stable maximal tori of G and the $(G^*)^{F^*}$-conjugacy classes of F^*-stable maximal tori of G^* ([2, Proposition 4.3.4]). This bijection will be denoted by $T \leftrightarrow T^*$.

We denote by σ_G the $\mathbf{F}_q$-rank of G (see [4]) and by $W(T)$ the Weyl group of G with respect to a maximal torus T of G. Let $\varepsilon_G = (-1)^{\sigma_G}$. The following lemma can be easily shown from the definitions.

LEMMA 2.1. *If* $T \leftrightarrow T^*$ *then we have:*

(1) $\varepsilon_G \varepsilon_T = \varepsilon_{G^*} \varepsilon_{T^*}$
(2) $|W(T)^F| = |W(T^*)^{F^*}|$
(3) $|G^F : T^F| = |G^{*F^*} : T^{*F^*}|$.

For a pair (T, θ) of an F-stable maximal torus T of G and a linear character θ of T^F, Deligne-Lusztig[4] defined a virtual character R_T^θ of G^F using the ℓ-adic cohomology groups. In their famous paper [4], Deligne-Lusztig also defined the notion of geometric conjugacy classes of the pairs (T, θ). If χ is an irreducible character of G^F and $(\chi, R_T^\theta) \neq 0$ for some (T, θ) contained in a geometric conjugacy class κ then we say $\chi \in \kappa$. We will freely use the standard results on the Deligne-Lusztig characters R_T^θ and the geometric conjugacy classes given in [4].

An irreducible character of G^F is called a uniform character if it is a $\mathbf{C}$-linear combination of Deligne-Lusztig characters. The proof of the following character formula is similar to that of [4, Lemma 10.6].

LEMMA 2.2. *Let* χ *be an irreducible uniform character of* G^F *contained in a geometric conjugacy class* κ. *Then*

$$\chi = \sum_{\substack{(T,\theta)\in\kappa \\ \text{mod } G^F}} \frac{(\chi, R_T^\theta)}{(R_T^\theta, R_T^\theta)} R_T^\theta$$

where the sum extends over one representative (T,θ) in each G^F-orbit in κ.

We denote by $\chi \mapsto \chi^*$ the Curtis-Alvis-Kawanaka duality operation on the character ring of G^F.

THEOREM 2.3 ([3], [1] AND [6]). *The map $\chi \mapsto \chi^*$ induces an isometry on the space of class functions of G^F which has order 2. Thus if χ and ξ are characters of G^F then $(\chi^*)^* = \chi$ and $(\chi,\xi) = (\chi^*,\xi^*)$.*

Since we have

$$(R_T^\theta)^* = \varepsilon_G \varepsilon_T R_T^\theta$$

by [5], we get:

COROLLARY 2.4. *Let χ be as in Lemma 2.2. Then*

$$\chi^* = \sum_{\substack{(T,\theta)\in\kappa \\ \mathrm{mod}\ G^F}} \frac{\varepsilon_G \varepsilon_T (\chi, R_T^\theta)}{(R_T^\theta, R_T^\theta)} R_T^\theta$$

where the sum extends over one representative (T,θ) in each G^F-orbit in κ.

3. The Operation $\chi \mapsto \chi_{\mathbf{u}}$

Let (G,F) and (G^*,F^*) be as in the previous section. An irreducible character χ of G^F is called an irreducible unipotent character if $(\chi, R_T^1) \neq 0$ for some F-stable maximal torus T of G. Thus the irreducible unipotent characters form a single geometric conjugacy class. This geometric conjugacy class is consisting of $(T,1)$ for F-stable maximal tori T, since $(T,1)$ is geometrically conjugate to (T',θ') if and only if $\theta'=1$. Now Proposition 2.2 implies that if χ is an irreducible unipotent uniform character of G^F then

$$(*) \qquad \chi = \sum_{\substack{T \\ \mathrm{mod}\ G^F}} \frac{(\chi, R_T^1)}{(R_T^1, R_T^1)} R_T^1$$

where the sum extends over G^F-conjugacy classes of F-stable maximal tori T.

DEFINITION 3.1. Let χ be an irreducible unipotent character of G^F. Define a class function χ_u of $(G^*)^{F^*}$ by

$$\chi_u = \sum_{\substack{T^* \\ \text{mod } (G^*)^{F^*}}} \frac{\mu^{\chi}_{T^*}}{|W(T^*)^{F^*}|} R^1_{T^*}$$

where $\mu^{\chi}_{T^*} = (\chi, R^1_T)$ if $T \leftrightarrow T^*$.

For example, we have $(1_G)_u = 1_{(G^*)}$ and $(St_G)_u = St_{(G^*)}$ where 1_G is the principal character of G^F and St_G is the Steinberg character of G^F.

It follows from [4, Theorem 6.8] that (R^1_T, R^1_T) is equal to $|W(T)^F|$, hence equal to $|W(T^*)^{F^*}|$ if $T \leftrightarrow T^*$ by Lemma 2.1. Let χ be an irreducible unipotent uniform character of G^F. Then $(*)$ and [4, Theorem 6.8] imply

$$(\chi_u, \chi_u) = \sum_{\substack{T^* \\ \text{mod } (G^*)^{F^*}}} \frac{(\mu^{\chi}_{T^*})^2}{|W(T^*)^{F^*}|} = \sum_{\substack{T \\ \text{mod } G^F}} \frac{(\chi, R^1_T)^2}{|W(T)^F|} = (\chi, \chi) = 1.$$

Moreover, since

$$R^1_T(1) = \varepsilon_G \varepsilon_T |G^F : T^F|_{p'}$$

by [4, Theorem 7.1], Lemma 2.1 and $(*)$ imply $\chi(1) = \chi_u(1)$. Thus χ_u is an irreducible unipotent uniform character of $(G^*)^{F^*}$.

Being inspired by the above argument, we make the following definition.

DEFINITION 3.2. We say G^F is unipotently uniform if all the irreducible unipotent characters of G^F are uniform characters.

Henceforth, we assume G^F is unipotently uniform. G^F is unipotently uniform if, for example, G^F is a split group of type A^{ad}_ℓ.

If χ and ξ are irreducible unipotent characters of G^F, then [4, Theorem 6.8], Lemma 2.1 and $(*)$ imply $(\chi_u, \xi_u) = (\chi, \xi)$. Furthermore, if $\chi \neq \xi$ then $(\chi_u, \xi_u) = (\chi, \xi) = 0$. This shows $\chi \mapsto \chi_u$ is injective and the next theorem follows from the fact that the number of irreducible unipotent characters of G^F is the same as the number of irreducible unipotent characters of $(G^*)^{F^*}$.

THEOREM 3.3. *Let G^F be unipotently uniform. Then the operation $\chi \mapsto \chi_u$ gives a bijection (in fact, an isometry) between the set of irreducible unipotent characters of G^F and the set of irreducible unipotent characters of $(G^*)^{F^*}$.*

Note that we have also shown, incidentally, that if G^F is unipotently uniform then so is $(G^*)^{F^*}$.

It is clear that the dual of (G^*, F^*) is (G, F). The following facts can be now easily verified using Corollary 2.4.

PROPOSITION 3.4. *Let G^F be unipotently uniform and χ be an irreducible unipotent character of G^F. Then we have:*

(1) $\chi(1) = \chi_u(1)$ *and* $(\chi, R_T^1) = (\chi_u, R_{T^*}^1)$ *if* $T \leftrightarrow T^*$

(2) $(\chi_u)_u = \chi$

(3) $(\chi^*)_u = (\chi_u)^*$.

Property (1) in the above proposition is also satisfied by the Lusztig's correspondence (see Section 1) by [7]. In fact, $(*)$ implies:

COROLLARY 3.5. *The correspondence $\chi \leftrightarrow \chi_u$ in Theorem 3.3 coincides with the Lusztig's correpondence.*

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