Comm. Korean Math. Soc. 6(1991), No. 2, pp. 257–260

SEPARABLE TWISTED GROUP ALGEBRAS

YOUNG SOO PARK AND EUN SUP KIM

1. Introduction

A very satisfactory characterization of group rings which are Azumaya algebras has been obtained by DeMeyer and Janusz in [3]. That is, KG is an Azumaya algebra if and only if $[G; \zeta(G)] < \infty$ and G' is finite with $|G'|^{-1} \in K$. However, the twisted group ring case does not seem to have an analogous characterization in general. Hence we have a question : when is a twisted group algebra an Azumaya algebra ?

In this paper, we give a partial solution of it, which generalizes the result of DeMeyer and Janusz [3]. In particular, we show that if α is a symmetric cocycle in $\zeta(G)$ and $[G : \zeta(G)] < \infty$ with $[G : \zeta(G)]^{-1} \in F$, then $F^\alpha G$ is an Azumaya algebra. Moreover, if G is a torsion-free, hypercentral group and $F^\alpha G$ is an Azumaya algebra, then $[G : \zeta(G)] < \infty$. G will denote a group, $\zeta(G)$ the center of G, K a commutative ring with 1, K^* the group of units of K and F a field. We recall that a ring is an Azumaya algebra if it is separable over its center.

2. Main results

It is easy to verify the next lemma.

LEMMA 1. *Let K and L be integral domains with $K \subset L$ and let $\alpha \in Z^2(G, K^*)$. Then*
(1) $L \otimes_K K^\alpha G \cong L^\alpha G$ as L-algebras
(2) $\zeta(L^\alpha G) \cong L \otimes_K \zeta(K^\alpha G)$
(3) $L^\alpha G = \zeta(L^\alpha G) \otimes_{\zeta(K^\alpha G)} K^\alpha G$ as L-algebras.

Received May 3,1991.
The authors acknowledge the support of both Korean Science and Engineering Foundation, and the Basic Science Research Institute Program, Ministry of Education 1990.

THEOREM 2. *Let F and L be fields with $F \subseteq L$ and let $\alpha \in Z^2(G, F^*)$. Then $F^\alpha G$ is an Azumaya algebra if and only if $L^\alpha G$ is an Azumaya algebra.*

Proof. Suppose $L^\alpha G$ is an Azumaya algebra. Since

$$\begin{aligned}\zeta(L^\alpha G) &= L \otimes_F \zeta(F^\alpha G)\\ &\cong (\amalg F) \otimes_F \zeta(F^\alpha G)\\ &\cong \amalg(F \otimes_F \zeta(F^\alpha G))\\ &\cong \amalg\zeta(F^\alpha G),\end{aligned}$$

$\zeta(F^\alpha G)$ is a direct summand of $\zeta(L^\alpha G)$ as $\zeta(F^\alpha G)$–module.

By Lemma 1 and Corollary 1.10 of [2], $F^\alpha G$ is an Azumaya algebra.

Conversely, suppose $F^\alpha G$ is an Azumaya algebra. $L^\alpha G = \zeta(L^\alpha G) \otimes_{\zeta(F^\alpha G)} F^\alpha G$ is an Azumaya algebra.

THEOREM 3. *Let $G = \zeta(G)H$ for some subgroup H of G and let α be a symmetric cocycle in $\zeta(G)$.*

Then $F^\alpha G$ is an Azumaya algebra if and only if $F^\alpha H$ is an Azumaya algebra.

Proof. Let T be a set of elements in $\zeta(G)$ which represent the cosets of H in G. Then any element g in G has a unique representation $g = ht$ with $t \in T$, $h \in H$.

Since α is symmetric, if g is an α–regular element of G and the α–regular class sum of g is finite, then h is an α–regular element of H and the α–regular class sum of h is finite. Thus

$$\zeta(F^\alpha G) = \oplus \sum_{t \in T} \zeta(F^\alpha H)\bar{t}.$$

Since

$$F^\alpha G = \oplus \sum_{t \in T} F^\alpha H \bar{t},$$

$$F^\alpha G = \zeta(F^\alpha G) \otimes_{\zeta(F^\alpha H)} F^\alpha H.$$

Now since $\zeta(F^\alpha H)$ is a direct summand of $\zeta(F^\alpha G)$, the equivalence stated in the theorem follows from Corollary 1.7 and Corollary 1.10 of [2].

THEOREM 4. *If* $[G:\zeta(G)]<\infty, [G:\zeta(G)]^{-1}\in F$ *and* α *is a symmetric cocycle in* $\zeta(G)$, *then* $F^\alpha G$ *is an Azumaya algebra.*

Proof. It suffices to show that there exists a separability idempotent e in $F^\alpha G\otimes_{\zeta(F^\alpha G)}(F^\alpha G)^\circ$. Let T be a transversal for $\zeta(G)$ in G. Let $e=[G:\zeta(G)]^{-1}\sum\bar{t}\otimes\bar{t}^{-1}$. Then it is sufficient to show $(\bar{x}\otimes 1)e=(1\otimes\bar{x})e$ for all $x\in G$. For each t in T, there is $c\in\zeta(G)$, $s\in T$ such that $xt=sc$.

Since $\bar{x}\bar{t}=\alpha(x,t)\alpha(s,c)^{-1}\bar{s}\bar{c}$ and α is symmetric,

$$\begin{aligned}(\bar{x}\otimes 1)(\bar{t}\otimes\bar{t}^{-1}) &= \bar{x}\bar{t}\otimes\bar{t}^{-1}\\ &= \alpha(x,t)\alpha(s,c)^{-1}\bar{s}\bar{c}\otimes\bar{t}^{-1}\\ &= \bar{s}\otimes\alpha(x,t)\alpha(s,c)^{-1}\bar{c}\bar{t}^{-1}\\ &= \bar{s}\otimes\bar{s}^{-1}\bar{x}\\ &= \bar{s}\otimes\bar{x}\bar{s}^{-1}\\ &= (1\otimes\bar{x})(\bar{s}\otimes\bar{s}^{-1}).\end{aligned}$$

As t runs through the elements of T, so does s. Thus $F^\alpha G$ is an Azumaya algebra.

A group G is said to be hypercentral if every nontrivial factor group of G has nontrivial center. One example of a hypercentral group is of course a nilpotent group. In fact, a finitely generated hypercentral group is nilpotent by a result of Mal'cev [6].

LEMMA 5 ([b]). *If* G *is a torsion-free hypercentral group, then* $\Delta(G)$ *is the center of* G.

THEOREM 6. *If* G *is a torsion-free hypercentral group and* $F^\alpha G$ *is an Azumaya algebra, then the center* $\zeta(G)$ *has finite index in* G.

Proof. Let $F^\alpha G$ be an Azumaya algebra. Then $F^\alpha G$ is a finitely generated projective $\zeta(F^\alpha G)$–module. Since the α–regular class sums $c(\bar{g})$ span $\zeta(F^\alpha G)$ as g runs through Δ, $\zeta(F^\alpha G)$ is a subring of $F^\alpha\Delta$. Let T be a transversal for Δ in G. Then $F^\alpha G=\oplus\sum F^\alpha\Delta\bar{t}$. Since $F^\alpha G$ is finitely generated over a subring of $F^\alpha\Delta$, it follows that the group G/Δ is finite. From Lemma 4, $G/\zeta(G)$ is finite.

References

1. G. Azumaya, *Separable rings*, J. Alg. **63** (1980), 1–14.
2. F.R. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, vol. 181, Lecture Notes in Math., Springer, Berlin and New York, 1971.
3. F.R. DeMeyer and G.J. Janusz,, *Group rings which are Azumaya algebra*, Trans. Amer. Math. Soc. **279** (1983), 389–395.
4. G.J. Janusz, *Separable algebras over commutative rings*, Trans. Amer. Math. Soc. **122** (1966), 416–479.
5. G. Karpilovsky, *Projective representations of finite groups*, Marcel Dekker, Inc., 1985.
6. D.H. McLain, *Remarks on the upper central series of a group*, Proc. Glasgow Math. Assoc. **3** (1956), 38–44.
7. Y.S. Park and E.S. Kim, *The center of twisted group algebras over connected rings*, Math. Japonica **36** (1991), 977–981.

Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea