

ON PROJECTIVE MAPPING OF RECURRENT FINSLER SPACES

HONG SUH PARK AND BYUNG DOO KIM

0. Introduction

A diffeomorphism of a Finsler space F^n to another Finsler space \bar{F}^n is called a projective mapping if it maps an arbitrary geodesic on a geodesic. Projective mappings in Finsler spaces have been studied by several authors([1],[2],[3],[4],[5]). Under the projective mapping, the projective deviation tensor is invariant and the curvature tensor of Berwald is also invariant if it satisfies a condition $Q_i = 0$.

The purpose of the present paper is to consider projective mappings between the recurrent Finsler spaces. After having introduced the main relations about projective mappings, in Theorem 3, we have the relations between a recurrent Finsler space F^n with the recurrence vector K_m and another recurrent Finsler space \bar{F}^n with the recurrence vector \bar{K}_m .

1. The Berwald connection and projective mapping

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space with the fundamental function $L(x, y)$ homogeneous of the first degree in y , where x is a point of M^n and y is an element of support. The metric tensor of F^n is given by

$$g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_j \dot{\partial}_i L^2(x, y),$$

where we put $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The equation of a geodesic in F^n is written as

$$(1.1) \quad \frac{dy^i}{dt} + \gamma_j^i y^j y^k = \alpha y^i, \quad \left(\frac{dx^i}{dt} = y^i \right)$$

Received April 29, 1991.

with respect to a general parameter t , where γ_j^i are Christoffel symbols constructed from g_{ij} . Throughout the present paper we shall use the usual index 0 to denote the contraction with the element of support y^i . Putting

$$(1.2) \quad 2G^i = \gamma_0^i, \quad G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j,$$

we can define the Berwald connection $B = (G^i_j, G^i_{kj})$ by

$$(1.3) \quad T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_k + T^m_j G^i_m k - T^i_m G^m_{jk},$$

for a Finsler tensor field $T^i_j(x, y)$, where $\partial_k = \frac{\partial}{\partial x^k}$.

Now, we consider two Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ on a common underlying manifold. As is well known, the mapping $L \rightarrow \bar{L}$ is a projective if and only if there exists a scalar field $p(x, y)$ on M^n such that

$$(1.4) \quad \bar{G}^i(x, y) = G^i(x, y) + p(x, y)y^i, \quad (p \neq 0).$$

The projective factor $p(x, y)$ is positively homogeneous function of degree 1 in y . From (1.4) we can readily obtain

$$(1.5) \quad \bar{G}^i_j = G^i_j + y^i p_j + \delta^i_j p,$$

$$(1.6) \quad \bar{G}^i_{jk} = G^i_{jk} + y^i p_{jk} + \delta^i_j p_k + \delta^i_k p_j,$$

where we put $p_j = \dot{\partial}_j p$ and $p_{jk} = \dot{\partial}_k p_j$.

On the other hand, the deviation tensor and the curvature tensor are given by

$$(1.7) \quad H^i_j = 2\dot{\partial}_j G^i - \partial_m G^i_j y^m + 2G^i_j G^m - G^i_m G^m_j,$$

$$(1.8) \quad H^i_{jk} = \frac{1}{3}(\dot{\partial}_j H^i_k - \dot{\partial}_k H^i_j), \quad H^i_{hk} = \dot{\partial}_h H^i_k,$$

which satisfy the identities

$$(1.9) \quad \begin{aligned} & (a) \sigma_{(hjk)} H^i_{hk} = 0, \\ & (b) \sigma_{(hjk)} H^i_{jk(h)} = 0, \\ & (c) \sigma_{(hjk)} \{ H^i_{mj(hj(k))} + G^i_{mr} H^r_{jk} \} = 0, \end{aligned}$$

where $G_h^i{}_{jk} = \dot{\partial}_h G_j^i{}_k$, and $\sigma_{(hjk)}$ means cyclic permutation of the indices h, j, k and summation.

Contracting $H_h^i{}_j$ and $H_h^i{}_{jk}$, we obtain

$$\begin{aligned}
 (1.10) \quad & \text{(a) } H_0^i{}_{jk} = H_j^i{}_k, \quad \text{(b) } H_0^i{}_k = H^i{}_k, \\
 & \text{(c) } H_{jk} = H_j^a{}_{ka} = \dot{\partial}_j H_k, \quad H_j^a{}_a = H_j, \\
 & \text{(d) } H_{0k} = H_k, \quad \text{(e) } H_0 = (n-1)H, \\
 & \text{(f) } H_a^a{}_{hj} = H_{jh} - H_{hj}, \quad \text{(g) } H_0^i = 0, \\
 & \text{(h) } H_j^i{}_k = -H_k^i{}_j, \quad \text{(i) } H_{i0} = (n-1)\dot{\partial}_i H - H_i.
 \end{aligned}$$

S.C.Rastogi discussed the properties of the projective factor $p(x, y)$ satisfying the condition $Q_i = p_{(i)} - pp_i = 0$ [4]. We shall investigate the meaning of $Q_i = 0$. In this case, the curvature tensor is invariant under the projective transformation (1.4), that is, $\bar{H}_h^i{}_{jk} = H_h^i{}_{jk}$.

S.C.Rastogi proved the following [4].

THEOREM A. *If $Q_i = 0$, then the scalar $p(x, y)$ and its derivatives satisfy the equations :*

$$\begin{aligned}
 (1.11) \quad & \text{(a) } p_r H_j^r{}_i = 0, \\
 & \text{(b) } p_{rk} H_j^r{}_i + p_r H_k^r{}_{ji} = 0, \\
 & \text{(c) } \sigma_{(ijk)} \{p_{rk} H_j^r{}_i\} = 0.
 \end{aligned}$$

Tranvectioning (1.11 a) by y^j and using (1.10 b), we have

$$(1.12) \quad H^r{}_i p_r = 0.$$

Differentiating (1.12) with respect to y^k and interchanging i and k , we have

$$(1.13) \quad H^r{}_i p_{rk} + H^r{}_k p_{ri} = 0,$$

by virtue of (1.8) and (1.11).

2. Projective mappings between recurrent Finsler spaces

A Finsler space F^n is called recurrent [6] if the curvature tensor $H_h^i{}_{jk}$ of F^n satisfies

$$(2.1) \quad H_h^i{}_{jk(m)} = K_m H_h^i{}_{jk},$$

where K_m is a non-zero covariant vector field and the curvature tensor does not vanish.

Let $B\bar{\Gamma} = (\bar{G}_k^i{}_j, \bar{G}^i{}_j)$ be the Berwald connection on the space $\bar{F}^n = (M^n, \bar{L})$ obtained from $F^n = (M^n, L)$ by a projective transformation (1.4). Then the covariant derivative of the curvature tensor in \bar{F}^n is given by

$$(2.2) \quad \begin{aligned} \bar{H}_h^i{}_{jk(\bar{m})} &= \partial_m \bar{H}_h^i{}_{jk} - \dot{\partial}_a \bar{H}_h^i{}_{jk} \bar{G}_m^a + \bar{H}_h^a{}_{jk} \bar{G}_a^i{}_m \\ &\quad - \bar{H}_a^i{}_{jk} \bar{G}_h^a{}_m - \bar{H}_h^i{}_{ak} \bar{G}_j^a{}_m - \bar{H}_h^i{}_{ja} \bar{G}_k^a{}_m. \end{aligned}$$

We assume that a recurrent space F^n is transformed into another recurrent space \bar{F}^n by a projective transformation. Using the fact that the curvature tensor satisfying $Q_i = 0$ is invariant under the projective mapping, and using (1.5) and (1.6), we have from (2.2)

$$(2.3) \quad \begin{aligned} (K_m - \bar{K}_m) H_h^i{}_{jk} - \dot{\partial}_a H_h^i{}_{jk} (p \delta_m^a + p_m y^a) + H_h^a{}_{jk} (y^i p_{am} \\ + \delta_a^i p_m + \delta_m^i p_a) - H_a^i{}_{jk} (y^a p_{hm} + \delta_h^a p_m + \delta_m^a p_h) \\ - H_h^i{}_{ak} (y^a p_{jm} + \delta_j^a p_m + \delta_m^a p_j) - H_h^i{}_{ja} (y^a p_{km} \\ + \delta_k^a p_m + \delta_m^a p_k) = 0. \end{aligned}$$

where \bar{K}_m is a recurrence vector in \bar{F}^n .

Now, we shall see how the recurrence vectors K_m and \bar{K}_m change by the projective transformation. First, we shall discuss the case of $K_m = \bar{K}_m$, that is, the recurrence vector of the F^n and the \bar{F}^n are same.

(i) The case of $K_m = \bar{K}_m$

Since $p(x, y)$ and $H_h^i{}_{jk}(x, y)$ are homogeneous of degree 1 and degree 0 in y respectively, we find

$$(2.4) \quad p_m y^m = 0, \quad p_{mi} p^i = 0, \quad \dot{\partial}_m H_h^i{}_{jk} y^m = 0.$$

Thus, transvectiong (2.3) with y^m and y^h , we have

$$(2.5) \quad -3pH_j^i{}_k + H^i{}_j P_k - H^i{}_k p_j = 0.$$

Transvecting this with y^j , we obtain

$$(2.6) \quad pH^i{}_k = 0,$$

which is a contradiction. In fact, since the curvature tensor does not vanish, the deviation tensor $H^i{}_k$ also does not vanish owing to (1.8). Hence, under the non-trivial transformation (1.4) the space does not have the same recurrence vector.

THEOREM 1. *If a recurrent Finsler space F^n is transformed into another recurrent one \bar{F}^n by a projective transformation (1.4) satisfying $Q_i = 0$, then the F^n and \bar{F}^n do not have the same recurrence vector.*

Next, we assume that the F^n and \bar{F}^n are both recurrent spaces with the different recurrence vectors.

(ii) **The casse $K_m \neq \bar{K}_m$**

By the similar method above, transvecting (2.3) with y^m and y^h , we find

$$(2.7) \quad (K_0 - \bar{K}_0)H_j^i{}_k - 3pH_j^i{}_k + H^i{}_j p_k - H^i{}_k p_j = 0.$$

From (1.10 g) we obtain

$$(2.8) \quad (K_0 - \bar{K}_0 - 4p)H^i{}_k = 0.$$

Since the deviation tensor does not vanish, we have

LEMMA 2. *If a recurrent space F^n is transformed into another recurrent one \bar{F}^n with the different recurrence vectors respectively, then we have*

$$(2.9) \quad K_0 - \bar{K}_0 = 4p.$$

We consider the case (2.9). Substituting (2.9) into (2.7), we get

$$pH_j^i{}_k = H^i{}_k p_j - H^i{}_j p_k.$$

Differentiating this equation with respect to y^l and cyclic permutation of the indices l, j, k and summation, we have

$$(2.10) \quad \sigma_{(jkl)} H_j^i p_l = 0,$$

by virtue of (1.8). Contracting (2.10) with respect to i and k and using (1.10), we obtain

$$(2.11) \quad H_j p_l - H_l p_j = 0.$$

Transvecting (2.11) with y^j , we get

$$(2.12) \quad (n - 1) H p_l - p H_l = 0,$$

where the scalar curvature H is non-zero.

On the other hand, transvecting (2.3) with y^h and cyclic permutation of the indices m, j, k and summation, we have

$$(2.13) \quad \sigma_{(mjk)} (K_m - \bar{K}_m) H_j^i p_k = 0.$$

Transvecting this equation with y^m and using (2.9), we get

$$(2.14) \quad 4p H_j^i p_k - (K_j - \bar{K}_j) H^i p_k + (K_k - \bar{K}_k) H^i p_j = 0.$$

Next, contracting (2.14) with respect to i and k and using (1.10 c), we have

$$(2.15) \quad (K_m - \bar{K}_m) H^m_j = (n - 1)(K_j - \bar{K}_j) H - 4p H_j.$$

Moreover, transvecting (2.3) with y^h and y^j and using (1.10), (1.11) and (2.4), we have

$$(2.16) \quad (K_m - \bar{K}_m) H^i_k + H^a_k p_{am} y^i - p H^i_{mjk} y^j - p H^i_{mk} - H^i_k p_m - H^i_m p_k = 0,$$

from which

$$(2.17) \quad (K_m - \bar{K}_m) H^m_k = p H^r_{rk},$$

in view of (2.12). From (1.10 f) and (1.10 i), (2.17) may be written as

$$(2.18) \quad (K_m - \bar{K}_m) H^m_j = p \{ (n - 1) \dot{\partial}_j H - 2H_k \}.$$

On substituting (2.15) in (2.18), from (2.12) we have

$$(2.19) \quad \begin{aligned} K_j - \bar{K}_j &= p(\dot{\partial}_j H / H) + 2p_j \\ &= p \dot{\partial}_j (\log H p^2). \end{aligned}$$

This shows the relation of two recurrence vectors under the projective mapping (1.4). Hence we have

THEOREM 3. *If a recurrent space F^n with the recurrence vector K_m is transformed into another recurrent one \bar{F}^n with the recurrence vector \bar{K}_m by a projective transformation (1.4) satisfying $Q_i = 0$, then we have*

$$K_m - \bar{K}_m = p\dot{\partial}_m(\log Hp^2).$$

By virtue of (2.15) and (2.12), we can easily prove the following theorem.

THEOREM 4. *If a recurrent space F^n with the recurrence vector K_m is transformed into another recurrent one \bar{F}^n with the recurrence vector \bar{K}_m by a projective transformation (1.4) satisfying $Q_i = 0$, then the following equations are equivalent*

- (a) $K_m - \bar{K}_m = 4p_m$,
 (b) $(K_a - \bar{K}_a)H_m^a = 0$.

References

1. M. Fuki and T. Yamada, *On Projective mappings in Finsler geometry*, Tensor N.S. **35** (1981), 216–222.
2. M. Mathumoto, *Projective change of Finsler metrics and projectively flat Finsler space*, Tensor N.S. **34** (1980), 303–315.
3. R.B. Misra, *The projective transformation in a Finsler space*, Ann. Soc. Sci Bruxelles **80** (1966), 227–239.
4. S.C. Rastogi, *On projective mappings in a Finsler space*, Tensor N.S. **42** (1985), 121–125.
5. H. Rund, *The differential geometry of Finsler spaces*, Springer, Berlin Göttingen Heidelberg, 1959.
6. B.B. Sinha and S.P. Singh, *On recurrent Finsler spaces*, Rev. Roum. Math. Pures et Appl. Tome XVI **6** (1971), 977–986.
7. P. Venzi, *On conformally recurrent manifolds*, Tensor N.S. **33** (1979), 40–42.

Department of Mathematics
 Yeungnam University
 Gyongsan 713-749, Korea