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ON SEMIORDERINGS OF LEVEL 2n

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1. Introduction

It is said that the idea of (quadratic) semiorderings is based back to Baer in an appendix of his book Linear Algebra and Projective Geometry [1]. A semiordering is well thought to be a certain generalization of positive cones and orders (cf. orderings) [2,9]. Prestel riveived this important notation and studied the relationship between semiorderings and valuations [11,12]. These semiorderings are generalized to semiorderings of level 2^n, furthermore to T-semiorderings where T is a preordering of level 2^n [2]. In this paper, we generalized above terminologies to semiorderings of level $2n$, furthermore to T-semiorderings where T is a preordering of level $2n$ and investigate the relationship as possible between these notions (semiorderings of level $2n$, T-semiorderings) and valuations by using the notion of compatibility.

2. Preliminaries

By a semiordering S of level $2n$ of a field F, we mean a subset S of F satisfing the following properties :

(1) $S + S \subset S$

(2) $F^{2n} S \subset S$

(3) $S \cap -S = \{0\}$

(4) $S \cup -S = F$.

In view of (3), we see $\text{char}(F) = 0$. Furthermore S is said to be a normed semiordering of level $2n$ if $1 \in S$ and above properties hold. Especially S is said to be a quardratic semiordering (q-ordering) if $n = 1$ and $1 \in S$. As orderings [9], S determines a linear order $\leq$ on F by $a \leq b \Leftrightarrow b - a \in S$.

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By a valuation on a field F, we shall always mean a Krull valuation $v : F^* := F - \{0\} \to \Gamma$ onto a (totally) ordered abelian group Γ, satisfying the two axioms :

$$(1) \qquad v(xy) = v(x) + v(y) \text{ for any } x, y \in F^*,$$
$$(2) \qquad v(x+y) \geq \min\{v(x), v(y)\} \text{ for any } x, y, x+y \in F^*.$$

The value group Γ will always be written additively, unless it is stated otherwise. For a given valuation v on F, we can define following collection of associated objects :

$$A := \{x \in F : x = 0 \text{ or } v(x) \geq 0\} \text{ (the valuation ring of } v),$$
$$I := \{x \in F : x = 0\} \text{ or } v(x) > 0 \text{ the maximal ideal of } v),$$
$$U := A \setminus I \text{(the group of valuation units)},$$
$$\bar{F} := A/I \text{(the residue class field of } v).$$

Usually we work with one valuation at a time, so given $x \in A$ we shall simply write $\bar{x}$ for $x + I$, its image in the residue class field. In the same vein, we shall adopt the following; For any subset $T \subseteq F$, let $\bar{T}$ denote the image of $T \cap A$ in the residue field $\bar{F}$. We shall refer to $\bar{T}$ as the pushdown of T (along v) into $\bar{F}$.

A semiordering S of level $2n$ of F will be said to be compatible with a valuation v of F (or v compatible with S) if

$$0 < a \leq b \Rightarrow v(a) \geq v(b) \text{ for all } a, b \in F.$$

A symmetric subset A of F (*i.e.* $a \in A \Rightarrow -a \in A$) is called convex with respect to S if

$$x < y < z, \text{ where } x, z \in A \Rightarrow y \in A.$$

PROPOSITION 2.1. *Let v be a valuation of F and S be a semiordering of level $2n$. Consider the following statements :*

(1) *S is compatible with v*
(2) *A is convex with respect to S*
(3) *I is convex with respect to S*
(4) $0 \leq a \in I \Rightarrow a < 1$.

Then (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) hold.

Proof. Suppose S is compatible with v. If $x < y < z$, where $x, z \in A$, then $0 < y - x < z - x$ and so by (1), $v(y-x) \geq v(z-x) \geq 0$. Hence $y - x \in A$ implies $y \in A$ and A is convex with repect to S. This means (1) $\Rightarrow$ (2). Suppose $a \geq 1$. Then 1 is contained in I. It is a contradiction. So $a < 1$ and this means (3) $\Rightarrow$ (4).

Let S be a normed semiordering of level $2n$. Setting

$$A(S) = \{a \in F : -k \leq a \leq k \text{ for some } k \in N\}$$

In [3], Becker proved that $A(S)$ is a valuation ring. Take $S^* = S - \{0\}$, for any nonempty subset S in a field. Then we get a following proposition.

PROPOSITION 2.2. *Let S be normed semiordering of level $2n$. Then*

$$S \cap U = \{a : a \in U, \bar{a} \in (\bar{S})^*\}.$$

Proof. Clearly $\subseteq$ holds. Consider $\bar{a} \in (\bar{S})^*$. Assume $a \notin S$, then $-a \in S$ and $-\bar{a} \in (\bar{S})$. Since $\bar{S}$ is an order [3], it is a contradiction.

PROPOSITION 2.3. *Let S be a normed semiordering of level $2n$. Then the familly Λ of valuation rings in F convex with respect to S forms a chain under inclusion. $A(S)$ is the smallest member in Λ.*

Proof. Suppose $A, B \in \Lambda$ but A is not a subset of B. By convexity of A, B, we have $B \subset A$.

So we have a following result.

COROLLARY 2.4. *Let S be a normed semiordering of level $2n$. If S is compatible with v, then $A \supset A(S)$.*

Proof. By proposition 2.1, it hold clearly.

PROPOSITION 2.5. *Let S be a normed semiordering of level $2n$ and v be a valuation F, where Γ is written additively. Consider the following statements :*

(1) *S is compatible with v.*

(1′) $(1+I)S \subseteq S$.
(2) *A is convex with respect to S.*
(2′) $(S \setminus A) + A \subseteq S$.
(3) *I is convex with respect to S.*
(3′) $(S \setminus I) + I \subseteq S$. *Then we can get (1) $\Longleftrightarrow$ (1′) $\Longrightarrow$ (2) $\Longleftrightarrow$ (2′) and (1) $\Longrightarrow$ (3) $\Longleftrightarrow$ (3′).*

Proof. Let $m \in I$ and $s \in S$. We want show that $(1-m)s \in S$. If not, $s < sm$ with $s \neq 0, m \neq 0$. So $v(s) \geq v(sm) = v(S) + v(m)$ and $v(m) \leq 0$ in Γ. It is a contradiction and this means (1) $\Rightarrow$ (1′). If $v(a) < v(b)$in Γ and $a \in S^* = S - \{0\}$. Then $\frac{b}{a}$ in I and so $a - b = a(1 - \frac{b}{a}) \in S^*(1+I) \subseteq S^*$ Hence $b < a$. This means (1) $\Leftarrow$ (1′). Others are similar.

Becker showed in [3] that for a valuation ring A and a normed semiordering of level $2n$ in a field F, A contains $A(S)$ if $1 + I \subset S$. Furthermore he proved that the following statements are equivelent:

(1) $(1+I)(U \cap S) \subset S$.
(2) $\bar{S}$ is a semiordering of level of $2n$ in A/I.
(3) $(1+I) \subset S$.

S is said to be weakly compatible with v if any one of them hold. We can have a following statement with respect to above.

PROPOSITION 2.6. *Let $\phi : A \longrightarrow A/I$ be the projection map and $\bar{S} = \phi(A \cap S)$ be the pushdown of normed semiordering of level $2n$. Then $\phi^{-1}((\bar{S})^*) \subset S$ is equivalent to above statements.*

Proof. Suppose $\phi^{-1}((\bar{S})^*) \subseteq S$. Let $x \in S \cap U$ and $y \in 1 + I$ then $\phi(xy) = xy + I = x + I \in (\bar{S})^*$ and $xy \in \phi^{-1}((\bar{S})^*) \subset S$. Conversely suppose that $(S \cap U)(1+I) \subset S$. Then S is a semordering of level $2n$ in A/I. Let $x \in \phi^{-1}((\bar{S})^*)$. This implies $x \in U$ with $\bar{x} \in (\bar{S})^*$. If $x \notin S$, then $\bar{x} \in -\bar{S}$. It is a contradiction. Therefore $x \in S$.

3. T-semiordering

By definition, a precordering T of level $2n$ on a field F is given by a subset $T \subset F$ with $-1 \notin T, T + T \subset T, F^{2n} \subset T$, and $TT \subset T$.

A nonempty set $M \subseteq F$ will be called a T-module if $M + M \subseteq M$ and $TM \subseteq M$ where T is a preordering of level $2n$. In particular, we always have $0 \in M$. In this case, $M \cap -M = \{0\}$ if and only if for $m_i \in M^* = M - \{0\}$ and $t_i \in T, t_1m_1 + t_2m_2 + \cdots + t_rm_r = 0$ implies $t_i = 0$. We shall say that M is an anisotropic T-module if the above holds. Let M be an anisotropic T-module and $a \in F^* = F - \{0\}$ in a formally real field [8], then $a \notin -M$ if and only if $M' := M + Ta$ is an anisotropic T-module. Furthermore in the case of anisotropic T-module $M \neq F$, M is maximally anisotropic if and only if $M \cup -M = F$. Therefore T-module $M \neq F$ which is maximally aisotropic is clearly a semiordering of level $2n$. Let T be a preordering of level $2n$. A subset S of F is called a T-semiordering if $S + S \subset S$, $TS \subset S$, $S \cap -S = \{0\}$ and $S \cap -S = F$. Since $F^{2n} \subset T, S$ is always a semiordering of level $2n$. Then above statement hold if S is a normed T-semiordering of level $2n$.

Let

$$A(S) = \{a \in F : -k \leq a \leq k \text{ for some } k \in N\}$$

and

$$I(S) = \{a \in F : \frac{-1}{k} \leq a \leq \frac{1}{k} \text{ for some } k \in N\}$$

For a normed semiordering S of higher level. This $A(S)$ is a valuation ring of F and $I(S)$ is the maximal ideal of $A(S)$ [3]. By computation $A(S)$ is convex. Furthermore we have

PROPOSTION 3.1. *For any normed T-semiordering S, there is a residually real valuation ring $A(S)$ with valuation v_s which is weakly compatible with S such that $\bar{S}$ is an order of $A(S)/I(S)$.*

Proof. Suppose $x < y < z$ with $x, z \in I(S)$. Since $A(S)$ is convex $I(S)$ is convex. By proposition 2.5 S is weakly compatible with v_s.

Let S be a normed T-module and v be any valuation compatible with S such that the pushdown $\bar{S}$ is an order [cf, proposition 2.2]. Since $TS \subset S$, we have $\bar{T} \subset \bar{S}$. Then we can form the wedge product $T \wedge \bar{S} = T\phi^{-1}((\bar{S})^*)$, which is a preordering of level $2n$ in F [5]. Furthermore, since v is compatible with S, I is convex with respect to S, which is equivalant to $(S \setminus I) + I \subseteq S$. So $1 + I \subset S$ (*i.e.*, weakly compatible), and we have $\phi^{-1}((\bar{S})^*) \subset S$ by proposition 2.6. Hence $T \wedge \bar{S} = T\phi^{-1}((\bar{S})^*) \subset S$.

LEMMA 3.2. *For $a \in S^*, v(a) \in v(T^*)$ if and only if $a \in T \wedge \bar{S}$.*

Proof. Suppose $v(a) \in v(T^*)$ then $v(a) = v(t)$ for some $t \in T$. Since $v(\frac{a}{t}) = 0, a = tu$ for some $u \in U$. So $u \in S^* \cap U \subset \phi^{-1}((\bar{S})^*)$. Hence $a \in T \wedge \bar{S}$. Conversely it hold similarly.

PROPOSITION 3.3. *Let S, v be as above such that $\bar{S}$ is an order and let $\{a_i; i \in I\} \subset S^*$ generates S as a T-modules. If $v(a_i) \in v(T^*)$ for all i, then S is preordering of level $2n$(Especially S is closed under multiplication)*

Proof. By above lemma, we have $a \in T \wedge \bar{S}$ for any i. So $S = \sum Ta_i, \subseteq T \wedge \bar{S} \subseteq S$. Hence $S = T \wedge \bar{S}$ and S is a preordering of level $2n$. (Especially S is closed under multiplication.)

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