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ORDER OF STARLIKENESS FOR MULTIPLIERS OF MEROMORPHIC UNIVALENT FUNCTIONS

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1. Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. Let $\sum_s$, $\sum^*(\alpha)$ and $\sum_k(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\sum$ that are univalent, meromorphically starlike of order α and meromorphically convex of order α, respectively. Analytically, f, of the form (1.1), is in $\sum^*(\alpha)$, if and only if

$$Re\{-\frac{zf'(z)}{f(z)}\} > \alpha \tag{1.2}$$

for $z \in U = \{z : |z| < 1\}$. Similarly, $f \in \sum_k(\alpha)$ if and only if f is of the form (1.1) and satisfies

$$Re\{-(1 + \frac{zf''(z)}{f'(z)})\} > \alpha \tag{1.3}$$

for $z \in U$.

The class $\sum^*(\alpha)$ and similar other classes have been extensively studied by Pommerenke[7], Clunie[2], Miller[5], Royster[8] and others.

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Let $\sum_p$ denote the class of functions of the form

$$(1.4) \qquad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \ a_n \geq 0,$$

that are regular and univalent in D and set $\sum_p^*(\alpha) = \sum_p \bigcap \sum^*(\alpha)$. It is further known [6] that a necessary and sufficient condition for a function to be in $\sum_p^*(\alpha)$ is that its coefficients satisfy

$$(1.5) \qquad \sum_{n=1}^{\infty} (n+\alpha) a_n \leq 1 - \alpha.$$

In [1], it was proved that the integral transform

$$(1.6) \qquad J(f) = \int_0^1 u f(uz) du$$

preserves meromorphically starlikeness (convexity). In particular, it was shown [6] that the integral transform takes meromorphically starlike functions to functions meromorphically starlike of order $\frac{1}{2}$.

In this paper, we introduce a general class that will incorporate most of the subclasses in [6, 10] and for which we determine extreme points, distortion properties, order of meromorphically starlikeness, radius of meromorphically convexity and other extremal properties.

DEFINITION. A function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$, is said to be in the class $\sum(\{b_n\})$ if there exists a sequence $\{b_n\}$ of positive real numbers such that $\sum_{n=1}^{\infty} b_n a_n \leq 1$.

2. Extremal properties

THEOREM 1. *$\sum(\{b_n\})$ is a convex class and, if $\sum(\{b_n\}) \subset \sum_p$, then $b_n \geq n$ for every n.*

Proof. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ are in $\sum(\{b_n\})$ and $0 \leq \lambda \leq 1$, then $\sum_{n=1}^{\infty} (\lambda a_n + (1-\lambda) c_n) b_n = \lambda \sum_{n=1}^{\infty} a_n b_n +$

$(1-\lambda)\sum_{n=1}^{\infty} c_n b_n \leq \lambda + (1-\lambda) = 1$ so that $\lambda f + (1-\lambda)g \in \sum(\{b_n\})$. Hence, $\sum(\{b_n\})$ is a convex class.

If $b_n < n$ for some n, then $f_n(z) = \frac{1}{z} + \frac{z^n}{b_n}$ has a derivative that vanishes at $z = (\frac{b_n}{n})^{\frac{1}{n+1}} < 1$ and $f_n(z) \in \sum(\{b_n\})$ is not univalent in D.

Let us write $\sum_p(\{b_n\}) = \sum_p \bigcap \sum(\{b_n\})$ where $\sum_p$ is the class of functions of the form (1.4) that are regular and univalent in D. Note that $\sum_p(\{\frac{n+\alpha}{1-\alpha}\}) = \sum_p^*(\alpha)$. We say that the order of meromorphically starlikeness of the class $\sum_p(\{b_n\})$ is α if $\sum_p(\{b_n\}) \subset \sum_p^*(\alpha)$ and $\sum_p(\{b_n\}) \not\subset \sum_p^*(\beta)$ for any $\beta > \alpha$.

THEOREM 2. *Let $f_o(z) = \frac{1}{z}$ and $f_n(z) = \frac{1}{z} + \frac{z^n}{b_n} (n = 1, 2, \cdots)$. Then $f \in \sum_p(\{b_n\})$ if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0 (n = 0, 1, 2, \cdots)$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.*

Proof. If $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (\frac{\lambda_n}{b_n}) z^n$, then $\sum_{n=1}^{\infty} b_n (\frac{\lambda_n}{b_n}) = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_o \leq 1$. Thus, $f \in \sum_p(\{b_n\})$. Conversely, if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \sum_p(\{b_n\})$, set $\lambda_n = b_n a_n$ $(n = 1, 2, \cdots)$ and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$.

COROLLARY 1. *The extreme points of $\sum_p(\{b_n\})$ are the functions $f_n(z) (n = 0, 1, 2, \cdots)$.*

COROLLARY 2. *If $f \in \sum_p(\{b_n\})$, $\{b_n\}$ increasing, then*

$$\frac{1}{r} - \frac{r}{b_1} = f_1(-r) \leq |f(z)| \leq f_1(r) = \frac{1}{r} + \frac{r}{b_1} \quad (|z| = r).$$

Proof. The extremal function must be one of the extreme points. But $f_1(-r) \leq |f_n(z)| \leq f_1(r)$ $(0 < |z| < r < 1)$.

3. Order of meromorphically starlikeness

By (1.5), the function $f_1(z) = \frac{1}{z} + \frac{z}{b_1}$ is in $\sum_p^*(\frac{b_1-1}{b_1+1})$. The next theorem gives a condition on $\{b_n\}$ for which $f_1(z)$ is extremal.

THEOREM 3. *If $b_n \geq \frac{(n+1)b_1+n-1}{2}$ for every n, then the order of meromorphically starlikeness of $\sum_p(\{b_n\})$ is $\frac{b_1-1}{b_1+1}$, with equality for $f_1(z) = \frac{1}{z} + \frac{z}{b_1}$.*

Proof. For $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ in $\sum_p(\{b_n\})$, it suffices to show that

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}+2(\frac{b_1-1}{b_1+1})-1}\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{2(1-\frac{b_1-1}{b_1+1})\frac{1}{z} - \sum_{n=1}^{\infty}(n+2(\frac{b_1-1}{b_1+1})-1)a_n z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{2(1-\frac{b_1-1}{b_1+1}) - \sum_{n=1}^{\infty}(n+2(\frac{b_1-1}{b_1+1})-1)a_n|z|^{n+1}} < 1.$$

Upon clearing the denominator in the last expression and letting $|z| \to 1$, we obtain

$$\sum_{n=1}^{\infty}(\frac{(n+1)b_1+n-1}{2})a_n \leq 1.$$

Since $b_n \geq \frac{(n+1)b_1+n-1}{2}$ and $\sum_{n=1}^{\infty} b_n a_n \leq 1$, the result follows.

COROLLARY 1. *If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \sum_p^*(\alpha)$, then the integral transform*

$$g(z) = c\int_0^1 u^c f(uz)du, \ 0 < c < \infty,$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{ca_n}{n+c+1}z^n \in \sum\nolimits_p^*(\frac{1+\alpha}{1+c+\alpha}).$$

Proof. Setting $b_n = \frac{(n+\alpha)(n+c+1)}{(1-\alpha)c}$, we see that $g \in \sum_p(\{b_n\})$. Since $\frac{b_1-1}{b_1+1} = \frac{1+\alpha}{1+c+\alpha}$, it suffices to show that $b_n \geq \frac{(n+1)b_1+n-1}{2}$, which is equivalent to $n^2 - 1 \geq 0$.

REMARK. When $c = 1$ and $\alpha = 0$, we obtain the results of [1,6].

COROLLARY 2. *Let* $b_n = (1-\delta)n + \delta n(n+1)$. *Then* $\sum_p(\{b_n\}) \subset \sum_p^*\left(\frac{1}{2+\alpha}\right)$ *for* $\delta \geq 0$, *and* $\sum_p(\{b_n\}) \not\subset \sum_p$ *for* $\delta < 0$.

Proof. Since $\frac{b_1-1}{b_1+1} = \frac{1}{2+\delta}$, Theorem 3 may be applied for $\delta \geq 0$ if $b_n \geq \frac{(n+1)b_1+n-1}{2}$, which is equivalent to $\delta(2n^2-n-1) \geq 0$. If $\delta < 0$, then $b_1 = 1+\delta < 1$, so we see from Theorem 1 that $\sum_p(\{b_n\}) \not\subset \sum_p$.

4. Other extremal functions

The next theorem gives a coefficient condition for which other extreme points represent the extremal function.

THEOREM 4. *If* $b_n \geq \frac{(n+1)b_k+n-k}{k+1}$ *for a fixed integer* k *and for every* n, *then the order of meromorphically starlikeness of* $\sum_p(\{b_n\})$ *is* $\frac{b_k-k}{b_k+1}$, *with equality for* $f_k(z) = \frac{1}{z} + \frac{z^k}{b_k}$.

Proof. By Theorem 2, we may write $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(\frac{\lambda_n}{b_n})z^n$ where $\sum_{n=1}^{\infty}\lambda_n \leq 1$. We must show, for $\alpha = \frac{b_k-k}{b_k+1}$, that $\sum_{n=1}^{\infty}\frac{(n+\alpha)\lambda_n}{(1-\alpha)b_n} \leq 1$. But $\frac{n+\alpha}{1-\alpha} = \frac{(n+1)b_k+n-k}{k+1} \leq b_n$ by hypothesis, so that $\sum_{n=1}^{\infty}\frac{(n+\alpha)\lambda_n}{(1-\alpha)b_n} \leq \sum_{n=1}^{\infty} \leq 1$, and the proof is complete.

COROLLARY. *If* $b_n > \frac{(n+1)b_k+n-k}{k+1}$ *for every* $n \neq k$, *then* $\sum_p(\{b_n\}) \subset \sum_p^*(\frac{b_k-k}{b_k+1})$, *with equality only for* $f_k(z) = \frac{1}{z} + \frac{z^k}{b_k}$.

Proof. It suffices to show, for $\alpha = \frac{b_k-k}{b_k+1}$, that $\sum_{n=1}^{\infty}(\frac{n+\alpha}{1-\alpha})\frac{\lambda_n}{b_n} < 1$ if $\lambda_k \neq 1$. Assume $\lambda_m > 0$ for some $m \neq k$. Since $b_m > \frac{m+\alpha}{1-\alpha}$ by hypothesis, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(\frac{n+\alpha}{1-\alpha})\frac{\lambda_n}{b_n} &= (\frac{m+\alpha}{1-\alpha})\frac{\lambda_m}{b_m} + \sum_{n\neq m}(\frac{n+\alpha}{1-\alpha})\frac{\lambda_n}{b_n} \\ &\leq (\frac{m+\alpha}{1-\alpha})\frac{\lambda_m}{b_m} + \sum_{n\neq m}\lambda_n \\ &< \lambda_m + \sum_{n\neq m}\lambda_n \leq 1.\end{aligned}$$

REMARK. Above Corollary shows that the function $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(\frac{\lambda_n}{b_n})z^n$ in $\sum_p(\{b_n\})$ is meromorphically starlike of order greater than $\alpha=\frac{b_k-k}{b_k+1}$, unless $\lambda_k=1$.

EXAMPLE. Let $b_n=2n,\ n\neq 2$ and $b_2=3$. Then $b_n>\frac{(n+1)b_2+n-2}{3}$ for $n\neq 2$, so that the order of meromorphically starlikeness of $\sum_p(\{b_n\})$ is $\frac{1}{4}$, with unique extremal function $f_2(z)=\frac{1}{z}+\frac{z^2}{3}$.

The final property we determine for the class $\sum_p(\{b_n\})$ is the radius of meromorphically converxity.

THEOREM 5. *If $f\in\sum_p(\{b_n\})$, then f is meromorphically convex in the disk*

$$|z|<\gamma_o=\inf_n\left(\frac{b_n}{n(n+2)}\right)^{\frac{1}{n+1}}\quad (n=1,2,\cdots).$$

The result is sharp, with extremal function of the form $f_n(z)=\frac{1}{z}+\frac{z^n}{b_n}$ for some n.

Proof. For $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(\frac{\lambda_n}{b_n})z^n$, it suffices to show that $|2+\frac{zf''(z)}{f'(z)}|\leq 1$ for $|z|\leq\gamma_o$. We have

$$|2+\frac{zf''(z)}{f'(z)}|=\left|\frac{\sum_{n=1}^{\infty}n(n+1)\frac{\lambda_n}{b_n}z^{n-1}}{\frac{1}{z}-\sum_{n=1}^{\infty}(\frac{n}{b_n})\lambda_n z^{n-1}}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty}n(n+1)(\frac{\lambda_n}{b_n})|z|^{n+1}}{1-\sum_{n=1}^{\infty}(\frac{n}{b_n})\lambda_n|z|^{n+1}},$$

which is bounded above by 1, if

$$(*)\qquad \sum_{n=1}^{\infty}n(n+2)\frac{\lambda_n}{b_n}|z|^{n+1}\leq 1.$$

Since $\sum_{n=1}^{\infty}\lambda_n\leq 1$, inequality $(*)$ is true for $|z|\leq\gamma_o$.

References

1. S. K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev, Roumanie Math. Pures Appl. **22** (1977), 295–297.
2. J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.
3. R. M. Goel and N. S. Sohi, *On a class of meromorphic functions*, Glasnik Matematicki **17** (1981), 19–28.
4. O. P. Juneja and M. L. Mogra, *On starlike functions of order α and type β*, Rev. Roumaine Math. Pures Appl. **23** (1978), 751–765.

J. E. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc. **25** (1970), 220–228.

6. M. L. Mogra, T. R. Reddy and O. P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. **32** (1985), 161–176.
7. Ch. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math. **13** (1963), 221–235.
8. W. C. Royster, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. **107** (1963), 300–308.
9. H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.
10. B. A. Uralegaddi and M. D. Ganigi, *A certain class of meromorphically starlike functions with positive coefficients*, Pure and Applied Math. Sci. **26** (1987), 75–81.

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