Comm. Korean Math. Soc. 6(1991), No. 2, pp. 213–224

BANACH SPACES OF THE TYPE Q_s

YOUNG SIK PARK

0. Introduction

This paper is to study the Banach spaces of analytic functions of the type Q_s. Spaces of the type Q'_s include as a special case, the space of Fourier ultrahyperfunctions and can be characterized by the Fourier transformation.

As far as we know the study of Fourier transforms and differential operators(local or nonlocal) has been focused for the inductive limit and projective limit of the Banach spaces $Q_s(T(K);K')$. For example, the spaces $Q(C^n)$ Park–Morimoto [8], $Q(R^n)$ Park [9], $\overleftarrow{Q_s}(T(U);U')$ and $\vec{Q}_s(T(K);K')$ Sargos–Morimoto [5] are studied for their Fourier transforms and differential operators(local or nonlocal) on them.

When we take inductive limit and projective limit, there were merits such that theorems for the spaces of them could be described beautifully. However, there were so many difficulties for the spaces $Q_s(T(K);K')$ to describe theorems.

In this paper, we formulate theorems for the Banach spaces $Q_s(T(K);K')$ elegantly to overcome such difficulties. Also we, as for limit, define the new space $Q_s(T(K);R^n)$ and inspect properties of the space.

In Sec.1, we give the main notations and necessary definitions. In Sec.2, we introduce and study spaces of the type Q_s and their duals Q'_s, in particular, the space of Fourier ultrahyperfunctions $Q'(C^n)$ described in Park–Morimoto [8]. In Sec.3, we show that the Fourier transfomation is defined and maps a space of the type Q_s into a space of the type Q_s; this enables us to define the Fourier transformation in spaces of the type Q'_s as well. In Sec.4, we give several results concerning the density of the space $\overleftarrow{Q_s}(C^n)$ in the spaces of the type Q_s. In Sec.5, we investigate

Received March 11, 1991.

the connection between entire functions of infra–exponential growth and differential operators(local or nonlocal) in spaces of the type Q_s.

1. Definitions and notations

We shall use the following notation:

$x, y, \xi, \eta, \cdots$ are points of the n–dimensional real space R^n with scalar product $x\xi$ and the Euclidean norm $\|z\| = \sqrt{xx}; z = x + iy, \zeta = \xi + i\eta, \cdots$ are points of the n–dimencional complex space C^n with bilinear form $z\zeta = (x\xi - y\eta) + i(x\eta + \xi y)$ and the Euclidean norm $\|x\| = \sqrt{xx + yy}$. $\bar{M}, \overset{\circ}{M}$, chM are the closure, interior, and convex hull of the set $M \subset R^n$.$M_1 \Subset M_2$ means that the set M_1 is bounded and $\bar{M}_1 \subset \overset{\circ}{M}_2$. $T(M) = R^n + iM = \{z = x + iy : y \epsilon M\}$ is the horizontal band in C^n over the set $M \subset R^n$. Let K, K', L, L' be convex compact sets in R^n with nonempty interior and U, U' be convex open sets in R^n.

We define the nega–support function W_M of the set $M \subset R^n$ by the equation

$$W_M(\xi) = inf\{\xi x : x \epsilon M\}.$$

We enumerate the most important properties of the nega–support function : for the region $M \subset R^n$,

(a) $W_M = W_{chM} = W_{\overset{\circ}{M}} = W_{\bar{M}}$;

(b) W_K is a continuous, concave, and homogeneous function in R^n;

(c) $W_{K+L} = W_K + W_L$;

(d)$W_L \leq W_K$ for every $K \subset L$;

(e) $W_L(\xi) + \varepsilon|\xi| \leq W_K(\xi)$, for every $K \Subset L$ there exists $\varepsilon > 0$;

(f) $W_K + W_{-L} \leq 0$ for every $K \subset L$;

(g) $W_K(\xi - \xi') \leq W_K(\xi) - W_K(\xi') \leq -W_{-K}(\xi - \xi')$.

We recall that the support function s_M of the set $M \subset R^n$ is defined by the equation $s_M(\xi) = \sup\{-y\xi : y \in M\}$. Hence we have $W_M = -s_M$.

We define the exponential e^{iz} as the function defined by the equation $e^{iz}(\zeta) = e^{iz\zeta}$; thus, e^{iz} is a function of the $\zeta(\in C^n)$ which dependes on the parameter $z(\in C^n)$.

Spaces of the type Q_s are spaces on which the exponentials e^{iz} determine the Fourier transformation $\mathcal{F}$ (by the equation $\mathcal{F}[\phi](z) = (e^{iz}, \phi) =$

$\int e^{iz(\xi+i\eta)}\phi(\xi+i\eta)d^n\xi$, ϕ is a test function) and in such a way that $\mathcal{F}$ maps a space of the type Q_s into a(possibly different)space of the type Q_s.

2. The space $\mathbf{Q_s(T(K);K')}$, and spaces of the type $\mathbf{Q_s}$

We denote by $Q_s(T(K);K')$ the Bananch space consisting of all functions ϕ which are continuous on $T(K)$ and holomorphic in the interior $T(\overset{\circ}{K})$ of $T(K)$ and have the finite norm

$$(2.1) \qquad \|\phi\|_{K,K'} = \sup\{e^{-W_{K'}(x)}|\phi(z)| : z = x+iy \in T(K)\},$$

The space $Q_s(T(K);K')$ is endowed with the topology determined by this norm(we recall once more that $K \subset R^n$ and $K' \subset R^n$ are convex compact sets with nonempty interior) (cf. [9]).

Suppose $L \subset K, L' \subset K'$, then obviously the restriction mapping defines the natural embedding mapping

$$(2.2) \qquad i^{KL}_{K'L'} : Q_s(T(K);K') \longrightarrow Q_s(T(L);L').$$

If $L \Subset K, L' \Subset K'$, the mapping $i^{KL}_{K'L'}$ is a compact mapping. The mapping $i^{KL}_{K'L'}$ enables us to construct inductive and projective limits of the spaces $Q_s(T(K);K')$. We define

$$(2.3) \qquad \vec{Q}_s(T(0);(0)) = \underset{\substack{\{0\}\Subset K \\ \{0\}\Subset K'}}{\text{ind lim}}\, Q_s(T(K);K')$$

We denote the space $\vec{Q}_s(T(0);(0))$ by $\vec{Q}_s(R^n)$. It is the space of infra–exponential real analytic functions which is the inductive limit of the spaces $Q_s(T(K);K')$ with respect to all pairs $K \Supset \{0\}, K' \Supset \{0\}$. Note that Kawai [10](resp. Zharinov [6]) uses for the space $\vec{Q}_s(R^n)$ the notation $Q(D^n)$ (resp. $\vec{\Phi}$). Obviously, we can restrict ourselves to the pairs $K = K' = \{\xi : |\xi| \leq \rho\} \equiv \bar{U}_\rho, \rho \to 0$. Thus, the function ϕ belongs to $\vec{Q}_s(R^n)$ if there exists $\rho > 0$ such that ϕ is continuous on $T(\bar{U}_\rho)$ and holomorphic in the interior $T(U_\rho)$ and that $|\phi(z)| \leq C\exp(-\rho\|x\|)$ for some constant $C > 0$.

Because the mapping $i_{K'L'}^{KL}$ is compact, the space $\vec{Q}_s(R^n)$ is of the type (DFS) with the locally convex inductive limit topology. Therefore, the dual space $\vec{Q}'_s(R^n)$ is the space of the type (FS), and

$$\vec{Q}'_s(R^n) = \operatorname*{proj\,lim}_{\substack{\{0\}\Subset K \\ \{0\}\Subset K'}} Q'_s(T(K);K') \tag{2.4}$$

is the projective limit of the Banach spaces $Q_s(T(K);K')$ dual to the corresponding spaces $Q_s(T(K);K')$.

We require the space

$$\overleftarrow{Q_s}(C^n) = \overleftarrow{Q_s}(T(R^n);R^n) = \operatorname*{proj\,lim}_{\substack{K\Subset R^n \\ K'\Subset R^n}} Q_s(T(K);K') \tag{2.5}$$

which is the projective limit of the spaces $Q_s(T(K);K')$ with respect to all pairs $K \Subset R^n, K' \Subset R^n$. Once more, it is sufficient to restrict ourselves to pairs $K = K' = \bar{U}_\rho, \rho \to \infty$. Thus, the function $\phi \in \overleftarrow{Q_s}(C^n)$ if it is entire and $\|\phi\|_{\bar{U}_\rho,\bar{U}_\rho} < \infty$ for all $\rho > 0$. Recall $\overleftarrow{Q_s}(C^n)$ is denoted by $Q(C^n)$ in [8]. Once more, since the mapping $i_{K'L'}^{KL}$ is compact, the space $\overleftarrow{Q_s}(C^n)$ is of type (FS), the dual space $\overleftarrow{Q_s}'(C^n)$ is of type (DFS) and

$$\overleftarrow{Q_s}'(C^n) = \operatorname*{ind\,lim}_{\substack{K\Subset R^n \\ K'\Subset R^n}} Q'_s(T(K);K') \tag{2.6}$$

is the inductive limit of the spaces $Q'_s(T(K);K')$. The elements of the dual space $\overleftarrow{Q_s}'(C^n)$ are called the Fourier ultrahyperfunctions in the Euclidean n–space(see [8]).

For convex compact sets L and L' with nonempty interior and for open convex sets U and U' of R^n, we put

$$\vec{Q}_s(T(L);L') = \operatorname*{ind\,lim}_{\substack{L\Subset K \\ L'\Subset K'}} Q_s(T(K);K'), \tag{2.7}$$

$$\overleftarrow{Q_s}(T(U);U') = \operatorname*{proj\,lim}_{\substack{K\Subset U \\ K'\Subset U'}} Q_s(T(K);K') \tag{2.8}$$

The spaces $Q_s(T(K);K')$ and the spaces obtained from them by means of the inductive and projective limits will be called here spaces of the type Q_s; their dual spaces will be called of the type Q'_s.

3. The Fourier transfomation on spaces of the type $\mathbf{Q_s}$ and $\mathbf{Q'_s}$

We define the Fourier tranform of a function $\phi \in Q_s(T(K);K')$ to be the function $\mathcal{F}[\phi]$ defined by

$$\mathcal{F}[\phi](z) = \int e^{iz\zeta}\phi(\zeta)d^n\xi = \int e^{iz(\xi+i\eta)}\phi(\xi+i\eta)d^n\xi,\ \eta \in K$$

(we emphasize that here η is fixed point in K). Elementary arguments show that for any $z = x+iy \in T(K')$ the integral on the right is defined and that its value does not depend on the choice of $\eta \in K$. Therefore, the function $\mathcal{F}[\phi]$ is defined in $T(K')$. Moreover, it is easy to show that for all $L \Subset K, L' \Subset K'$ we have

$$\mathcal{F}[\phi] \in Q_s(T(L');-L), \text{and } \|\mathcal{F}[\phi]\|_{L',-L} \le C\|\phi\|_{K,K'},$$

where the constant $C > 0$ does not depend on $\phi \in Q_s(T(K);K')$. Hence we have the following thorem:

THEOREM 3.1. *Suppose that L, K, L', K' are convex compact sets in R^n with nonempty interior and that $L \Subset K, L' \Subset K'$. Then the Fourier transformation $\mathcal{F}$ is a continuous linear mapping:*

$$Q_s(T(K);K') \longrightarrow Q_s(T(L');-L).$$

Suppose further $\psi = \mathcal{F}[\phi] \in Q_s(T(L');-L)$. Then the function ϕ can be uniquely recovered by means of the formula

$$\phi(\zeta) = \mathcal{F}^{-1}[\psi](\zeta) = \frac{1}{(2^\pi)^n}\int e^{-i\zeta(x+iy)}\psi(x+iy)d^nx, y \in L'.$$

Thus, $\mathcal{F}^{-1}[\psi](\zeta) = (2^\pi)^{-n}\mathcal{F}[\psi](-\zeta) \in Q_s(T(M);M')$ for all $M \Subset L \Subset K, M' \Subset L' \Subset K'$. Hence, for the inverse Fourier transformation, we have the followings:

THEOREM 3.2. *Suppose that M, L, M', L' are convex compact sets in R^n with nonempty interior and that $M \Subset L, M' \Subset L'$. Then the inverse Fourier transformotion $\mathcal{F}^{-1}$ is a continuous linear mapping:*

$$Q_s(T(L'); -L) \longrightarrow Q_s(T(M); M').$$

THEOREM 3.3. *Suppose that M, L, K, M', L', K' are convex compact sets in R^n with nonempty interior and that $M \Subset L \Subset K, M' \Subset L' \Subset K'$. Then the composite mapping $\mathcal{F}^{-1} \circ \mathcal{F}$ of*

$$\mathcal{F} : Q_s(T(K); K') \longrightarrow Q_s(T(L'); -L) \text{ and}$$
$$\mathcal{F}^{-1} : Q_s(T(L'); -L) \longrightarrow Q_s(T(M); M')$$

is equal to the canonical mapping

$$i^{KM}_{K'M'} : Q_s(T(K); K') \longrightarrow Q_s(T(M); M').$$

The Fourier transformation defined on $Q_s(T(K); K')$ can be transferred to the spaces $\vec{Q}_s(T(L); L')$ and $\overleftarrow{Q_s}(T(U); U')$, and we have the followings:

COROLLARY 1. *The Fourier transformation $\mathcal{F}$ gives topological isomorphisms:*

$$\mathcal{F} : \vec{Q}_s(T(L); L') \simeq \vec{Q}_s(T(L'); -L),$$
$$\mathcal{F}^{-1} : \overleftarrow{Q_s}(T(U); U') \simeq \overleftarrow{Q_s}(T(U'); -U).$$

The Fourier thansformation $\mathcal{F}$ introduced in the space $Q_s(T(K); K')$ generates in the dual $Q'_s(T(K); K')$ a dual transformation, which, following tradition, we shall also call a Fourier transformation and denote by $\mathcal{F}$. Namely, for $g \in Q'_s(T(K); K')$ we define $\mathcal{F}[g]$ by the equation

$$(\mathcal{F}[g], \phi) = (g, \mathcal{F}[\phi]), \text{for every } \phi \text{ such that } \mathcal{F}[\phi] \in Q_s(T(K); K').$$

Obviously, $\mathcal{F}[g] \in Q'_s(T(-J'); J)$ under the condition that $K \Subset J, K' \Subset J'$. Thus we have the following theorem:

THEOREM 3.4. *Suppose that* K, J, K', J' *are convex compact sets in* R^n *with nonempty interior and that* $K \Subset J, K' \Subset J'$. *Then the Fourier transfomation* $\mathcal{F}$ *is a continuous linear mapping:*

$$Q'_s(T(K); K') \longrightarrow Q'_s(T(-J'); J).$$

The Fourier transformation introduced on $Q'_s(T(K); K')$ can be transferred to the inductive and projective limits of such spaces. The inverse Fourier transfomation $\mathcal{F}^{-1}$ can also be defined in a natural manner.

THEOREM 3.5. *Suppose that* J, F, J', F' *are convex compact sets in* R^n *with nonempty interior and that* $J \Subset F, J' \Subset F'$. *Then the inverse Fourier transformation* $\mathcal{F}^{-1}$*is a continuous linear mapping:*

$$Q'_s(T(-J'); J) \longrightarrow Q'_s(T(F); F').$$

THEOREM 3.6. *Suppose that* F, J, K, F', J', K' *are convex compact sets with nonempty interior and that* $K \Subset J \Subset F, K' \Subset J' \Subset F'$. *Then the composite mapping* $\mathcal{F}^{-1} \circ \mathcal{F}$ *of*

$$\mathcal{F} : Q'_s(T(K); K') \longrightarrow Q'_s(T(-J); J),$$
$$\mathcal{F}^{-1} : Q'_s(T(-J'); J) \longrightarrow Q'_s(T(K); K')$$

is equal to the cononical mapping

$$i^{KK'}_{FF'} : Q'_s(T(K); K') \longrightarrow Q'_s(T(F); F').$$

COROLLARY 2. *The Fourier transformation* $\mathcal{F}$ *gives linear topological isomorphisms:*

$$\mathcal{F} : \vec{Q}'_s(T(L); L') \simeq \vec{Q}'_s(T(-L'); L),$$
$$\mathcal{F} : \overleftarrow{Q_s}(T(U); U') \simeq \overleftarrow{Q_s}(T(-U'); U).$$

In particular, the Fourier transformation defines topological isomorphisms of the spaces $\vec{Q}'_s(R^n)$ *and* $\overleftarrow{Q_s}'(C^n)$ *onto themselves.*

4. A number of results concerning the density of the space $\overleftarrow{Q_s}(\mathbf{C^n})$ in spaces of the type $\mathbf{Q_s}$

We denote by $Q_s(T(K);R^n)$ the set of all functions ϕ continous on $T(K)$ and holomorphic in the interior $T(\overset{\circ}{K})$ of $T(K)$, for which $\|\phi\|_{K,K'} < \infty$ for all convex compact sets $K' \subset R^n$ with nonempty interior.

PROPOSITION 4.1. *The space $Q_s(T(K);R^n)$ is the projective limit of the spaces $Q_s(T(K);K')$, where the projective limit is taken for all convex compact sets K' of R^n, that is,*

$$Q_s(T(K);R^n) = \operatorname*{proj\,lim}_{K' \Subset R^n} Q_s(T(K);K'). \tag{4.1}$$

We shall also assume everywhere that $L \Subset K$ and $L' \Subset K'$. Then we have the following relations:

$$Q_s(T(K);R^n) \subset Q_s(T(K);K') \subset Q_s(T(L);L') \tag{4.2}$$

PROPOSITION 4.2. *The space $Q_s(T(K);R^n)$ is dense in the space $Q_s(T(K);K')$ in the toplolgy of the space $Q_s(T(L);L')$.*

Proof. For every $\phi \in Q_s(T(K);K')$, we set $\phi_k(\zeta) = \exp(-k^{-l}\zeta^2)\phi(\zeta)$, where $\zeta^2 = \zeta\zeta$. It is easy to show that $\phi_k \in Q_s(T(K);R^n)$ for all $k = 1, 2, \cdots$ and $\|\phi - \phi_K\|_{L,L'}$ converges to 0 as $k \to \infty$, i.e., $\phi_k \to \phi$ in the topology of $Q_s(T(L);L')$.

We define

$$\vec{Q}_s(T(K);\{0\}) = \operatorname*{ind\,lim}_{\{0\} \Subset K'} Q_s(T(K);K'), \tag{4.3}$$

$$\vec{Q}_s(R^n;K') = \operatorname*{ind\,lim}_{\{0\} \Subset K'} Q_s(T(K);K') \tag{4.4}$$

Under the condition $\{0\} \Subset L \Subset K$, we have the following relations:

$$Q_s(T(K);K') \subset Q_s(T(L);L') \subset \vec{Q}_s(R^n;L') \tag{4.5}$$

PROPOSITION 4.3. *The space* $\overleftarrow{Q_s}(C^n)$ *is dense in the space* $\vec{Q}_s(R^n; K')$.

Proof. Put, for $\varepsilon > 0$,

$$g_\varepsilon(x) = (2\pi)^{-n/2}\varepsilon - n\exp(-(x_1^2 + \cdots + x_n^2)/2\varepsilon^2)$$

Then $g_\varepsilon(z)\varepsilon\overleftarrow{Q_s}(C^n)$. Let $\phi \in Q_s(R^n); K')$. As $\phi(\zeta)$ is decreasing when $|\xi| \to \infty(\zeta = \xi + i\varepsilon\eta)$, the convolutions

$$\begin{aligned} g_\varepsilon * \phi(\xi + i\eta) &= \int_{R^n} g_\varepsilon(x + iy)\phi((\xi + i\eta) - (x + iy))dx \\ &= \int_{R^n} g_\varepsilon(x_1 + iy_1, \cdots, x_n + iy_n)\phi((\xi_1 - x_1) + i(\eta_1 - y_1), \cdots, \\ &\qquad (\xi_n - x_n) + i(\eta_n - y_n))dx_1 \cdots dx_n \end{aligned}$$

are entire functions and $\|g_\varepsilon * \phi\|_{\bar{U}_\rho,\bar{U}_\rho} < \infty$ for all $\rho > 0$. Hence $g_\varepsilon * \phi \in \overleftarrow{Q_s}(C^n)$ for any $\varepsilon > 0$. On the other hand,

$$\|g_\varepsilon * \phi - \phi\|_{\bar{U}_\rho,K'} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \text{ for all } \rho > 0.$$

PROPOSITION 4.4. $\overleftarrow{Q_s}(C^n)$ *is dense in* $Q_s(T(K); R^n)$ *in the topology of* $Q_s(T(L); L')$.

This is obtained from the foregoing by means of a Fourier transformation.

PROPOSITION 4.5. $\overleftarrow{Q_s}(C^n)$ *is dense in* $Q_s(T(K); K')$ *in the topology of* $Q_s(T(L); L')$.

This follows from $\overleftarrow{Q_s}(C^n)$ being dense in $Q_s(T(K); R^n)$ and $Q_s(T(K); R^n)$ in $Q_s(T(K); K')$ (or $\overleftarrow{Q_s}(C^n)$ in $\vec{Q}_s(R^n; K')$ and $Q_s(T(K); K') \subset \vec{Q}_s(R^n; K')$).

$\overleftarrow{Q_s}(C^n)$ is dense in $\vec{Q}_s(R^n)$, which follows from the foregoing.

5. Entire functions of infra–exponential growth and differential operators(local or nonlcoal)

We shall denote by $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi–index;
$|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \cdots, z_n)$, $D^\alpha\phi(z) = \partial^{|\alpha|}\phi(z)/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$, *etc.*

DEFINITION [1]. An entire function $J(z)$ is said to be a function of $infra-exponential\ growth$ if for any $\varepsilon > 0$ there exists $C \geq 0$ such that

$$|J(z)| \leq C \exp(\epsilon\|z\|)$$

for all $z \in C^n$.

It is easy to show that if $J(Z)$ is an entire function of infra–exponential growth then operation of multiplication by $J(z)$ defines a continous linear mapping

$$J(z) : Q_s(T(L); K') \longrightarrow Q_s(T(L); L')$$

for every pair L, L' and K' with $L' \Subset K'$, and, therefore, $J(z)$ is a multiplier in the space $\vec{Q}_s(R^n)$, i.e.,

$$J(z) : \vec{Q}_s(R^n) \longrightarrow \vec{Q}_s(R^n)$$

is continuous and linear.

Suppose $J(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$ is the Taylor series expansion of an entire function $J(z)$.

The differential operator $J(-iD)$ is defined by the equation

$$J(-iD)\phi(\zeta) = \sum_{\alpha \geq 0} a_\alpha (-iD)^\alpha \phi(\zeta) = \sum_{\alpha \geq 0} a_\alpha (-i)^{|\alpha|} D^\alpha \phi(\zeta),$$

where ϕ is a sufficiently smooth function.

The diffrential operator $J(-iD)$ is said to be local if the function $J(z) = \sum_{\alpha \geq 0} a_\alpha \alpha! z^\alpha$ is entire, i.e., $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|a_\alpha| \alpha!} = 0$.

$J(-iD)$ is a local differential operator if and only if $J(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$ is an entire function of infra–exponential growth(see [1] or [6]).

THEOREM 5.1. *If $J(-iD)$ is a local operator then*

$$J(-iD) : Q_s(T(K); L') \longrightarrow Q_s(T(L); L')$$

is continuous and linear, for every pair L, L' and any $L \Subset K$. Therefore, we have the following:

COROLLARY 5.1. *If $J(-iD)$ is a local operator then*

$$J(-iD) : \vec{Q}_s(R^n) \longrightarrow \vec{Q}_s(R^n)$$

is a continuous linear differential operator.

If $J(-iD)$ is a local operator and $\phi \epsilon Q_s(T(K); K')$, then $\mathcal{F}[J(-iD)\phi](z) = J(z)\phi(z)$. Further, let $J(-iD)$ be a differential operator and g an analytic functional. The analytic functional $J(iD)_g$ is defined by the equation

$$(J(iD)_g, \phi) = (g, J(-iD)\phi).$$

If $J(-iD)$ is a local operator then the dual mapping is defined:

$$J(iD) : \vec{Q}'_s(R^n) \longrightarrow \vec{Q}'_s(R^n).$$

Suppose $J(z) = \exp(-az), a \in C^n$, then $J(-iD) = \exp(iaD)$ is a nonlocal differential operator. For every $\phi \in Q_s(T(K); K'), J(-iD)\phi(\zeta) = \phi(\zeta + ia)$. Hence we have the following:

THEOREM 5.2. *Suppose $J(z) = \exp(-az)$, $a \epsilon C^n$, then $J(-iD)$ is nonlocal and*

$$J(-iD) : Q_s(T(K); K') \longrightarrow Q_s(T(K + \{Re\ a\}_; K')$$

is a continuous linear differential operator.

THEOREM 5.3. *Suppose $J(z) = \sum_{j=1}^{n} \exp(-a_j z), a_j \in C^n$ and if Int $\cap(K + \{Re\ a_j\}) \neq \emptyset$, then $J(-iD)$ is nonlocal and*

$$J(-iD) : Q_s(T(K); K') \longrightarrow Q_s(T \bigcap_{j=1}^{n} (K + \{Re\ a_j\})); K')$$

is continuous and linear.

References

1. A. Kaneko, *On the structure of Fourier hyperfunctions*, Proc. Japan Acad. **48** (1972), 651–653.
2. A. Kaneko, *Representation of hyperfunctions by measures and some of its applications*, J. Fac. Sci., Univ. Toky Sec. IA **19** (1972), 321–352.
3. H. Komatsu, *Projective and injective limits of weakly compact sequences of locally convex spaces*, J. Math. Soci. Japan **19 (3) July** (1967), 366–383.
4. M. Morimoto, *Introduction to Sato Hyperfunctions*, Kyoritsu Shuppan Tokyo, 1976 (in Japanese).
5. P.Sargos and M. Morimot, *Transformation des Fonctionnelles Analytiques à Porteurs non compacts*, Tokyo J. of Math. **5 (2)** (1981), 457–492.
6. V.V. Zharinov, *Laplace Transformation of Fourier Hyperfunctions and Related classes of Analytic Functionals*, Teoret. i Mat. Fiz **33 (3)** (1977), 291–309.
7. V.V. Zharinov, *Distributive Lattices and Their Applicationa in complex Analysis*, Proc. Stek. Inst. Math.,, 1985 Issue 1.
8. Y.S. Park and M. Morimoto, *Fourier ultra-hyperfunctions in the Euclidean n-space*, J. Fac Sci. Univ. Tokyo Sec. IA Math. **20** (1973), 121–127.
9. Y.S. Park, *Fourier Ultra-Hyperfunctions Valued in a Fréchet Space*, Tokyo J. of Math **5 (1)** (1982), 143–155.
10. T.Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo Sec. IA Math. **17** (1970), 467–517.

Department of Mathematics
Pusan National University
Pusan 609–735, Korea