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ON QUASI-PERFECT RINGS AND SEMIHERDITARY MODULES

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1. Introduction

Throughout this paper a ring R is an associative ring with identity and all modules are unitary. Homomorphisms will be written on the right. The Jacobson radical will be denoted by J. A ring R is left hereditary if every left ideal of R is projective and left semihereditary if every finitely generated left ideal is projective. In [4], Hill introduced hereditary module, i.e. : A projective left module over a ring R is left hereditary if every submodule is projective. Using a result of Colby and Rutter [2] he proved that for P a finitely generated left hereditary module, $S = End_R(P)$ is left hereditary as a ring. And he also showed that a left perfect and left hereditary ring R is semiprimary.

In this article we deal with *C.P.* modules over quasi-perfect rings and endomorphism rings of semihereditary modules. Actually we show that a quasi–perfect left P.P. ring is semiprimary. Thereby we can generalize a result in [5]. We also obtain an analogous result for the endomorphism ring of a semihereditary module by using a method in [4] and a result of Small [5].

2. C.P. modules over quasi-perfect rings

A ring R is semilocal if R/J is semisimple Artinian. Recall the following characterization of left perfect rings due to Bass [1] : A ring R is left perfect iff R is semilocal and J is left T-nilpotent. Following Evans [3], we call a left R–module *C.P.* if every left cyclic submodule is projective.

In this section, the concept of a quasi–perfect ring is introduced. We begin by definition.

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DEFINITION 1. A ring R is quasi-perfect if R is semilocal and J is nil.

PROPOSITION 2. *Let R be a ring. Suppose Re is a C.P. module for each primitive idempotent e in R. Then the following are equivalent :*
(a) R is semiprimary.
(b) R is quasi-perfect.

Proof. (a) $\Longrightarrow$ (b) is clear by definition of semiprimary ring. (b) $\Longrightarrow$ (a) : Since J is nil, idempotents modulo J can be lifted. Thus R is semiperfect. Then $R = \sum_{i=1}^{n} Re_i$ where $\{e_1, \cdots, e_n\}$ is a set of primitive orthogonal idempotents whose sum is 1. We will show that J is nilpotent. Now $J = J(e_1 + \cdots + e_n) = Je_1 + \cdots + Je_n$, so it suffices to prove that Je_i is nilpotent for each $e_i, 1 \leq i \leq n$. Suppose there exists some Je_i which is not nilpotent. Then $(Je_i)^2 \neq 0$, so $e_i x e_i \neq 0$ for some $x \in J$. Consider the map $f_{xe_i} : Re_i \longrightarrow Re_i$ via $re_i \longmapsto re_i x e_i$. Since Re_i is C.P., Im f_{xe_i} is projective. Thus Ker $f_{xe_i} = 0$ for Re_i is indecomposable and Im $f_{xe_i} \neq 0$. Now $xe_i \in J$, so there exists an integer $m > 1$ such that $(xe_i)^m = 0$ and $(xe_i)^{m-1} \neq 0$. But $(xe_i)^{m-1} \in$ Ker $f_{xe_i} = 0$, a contradiction. Thus J is a finite sum of nilpotent left ideals and so is nilpotent. This result directly yields the following corollary.

COROLLARY 3. *Suppose R is quasi-perfect and left hereditary. Then R is semiprimary.*

Recall that a ring R is left P.P. if every principal left ideal of R is projective. We also recall a result due to Evans [3] : A ring R is left P.P. iff every projective left R-module is a C.P. module. Consequently we are able to extend Corollary 3 as follows.

COROLLARY 4. *Suppose R is quasi-perfect and left P.P. ring. Then R is semiprimary.*

3. Endomorphism rings of semihereditary modules

We begin by introducing definition.

DEFINITION 5. A left module over a ring R is called semihereditary if every finitely generated submodule is projective.

PROPOSITION 6. *Let R be a ring. If P_1 and P_2 are semihereditary R–module, then $P_1 \oplus P_2$ is semihereditary.*

Proof. Let N be finitely generated submodule of $P_1 \oplus P_2$. We will show that N is projective. Consider the projection map $\pi_1 : P_1 \oplus P_2 \longrightarrow P_1$. Clearly $\pi_1(N)$ is a finitely generated submodule of P_1. Since P_1 is semihereditary, $\pi_1(N)$ must be projective. So $N = (Ker\pi_1 \cap N) \oplus M$ where $M \cong \pi_1(N)$, since $0 \longrightarrow Ker\pi_1 \cap N \longrightarrow N \longrightarrow \pi_1(N) \longrightarrow 0$ is exact. Thus $Ker\pi_1 \cap N$ is a finitely generated submodule of P_2, and hence $Ker\pi_1 \cap N$ is projective. Hence N is Projective.

As a result, if P is semihereditary module, then $P^{(n)}$ is also semihereditary for all integers $n > 0$.

Recall a result due to Small [5] : A ring R is left semihereditary if and only if $Mat_n(R)$ is left P.P for all integers $n > 0$.

With minor modifications the same argument in [4] also serves to establish the next lemma.

LEMMA 7. *Let R be a ring and P a finitely generated left R–module. If P is semihereditary, then $S = End_R(P)$ is left P.P*

Proof. We will show that Sa is projective for every $a \in S$. Clearly $Q = Im(a)$ is a finitely generated submodule of the semihereditary module P. So Q must be projective, hence there exists a map $b : Q \longrightarrow P$ such that $ba = 1_Q$. Thus Qb is a direct summand of P. Now let $\pi : P \longrightarrow Qb$ be the natural projection. Define $f_b : Sa \longrightarrow S\pi$ via $sa \longmapsto sab$. Then f_b is well–defined S–homomorphism. It is easy to show that f_b is monomorphism. Moreover since P is projective, there exists a map $g : P \longrightarrow P$ such that $\pi = gab = (ga)f_b$. This implies that f_b is epimorphism. Thus $Sa \cong S\pi$. Since π is an idempotent, Sa is projective.

PROPOSITION 8. *Let R be a ring and P a finitely generated semihereditary left R–module. Then $S = End_R(P)$ is left semihereditary.*

Proof. By Lemma 7, $S = End_R(P)$ is a left P.P. ring and by proposition 6, $End_R(P^{(n)})$ is left P.P. for all integer $n > 0$. Now note that $Mat_n(S) \cong End_R(P^{(n)})$ as rings. So we have a semihereditary ring $S = End_R(P)$ by adapting Small's result in [5].

This result yields the following corollary due to Colby and Rutter [2].

COROLLARY 9. *Let R be left semihereditary and P a finitely generated projective left R-module. Then $S = End_R(P)$ is left semihereditary.*

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