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NOTES ON INFINITE LOOP SPACES AND DELOOPING MACHINE OF THE PLUS–CONSTRUCTION

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1. Introduction

J.F.Adam [1], R.J.May, S.B.Priddy, J.D.Stasheff and others [7], [9], [11] studied the infinite loop space intensively. In particular, M.G.Barratt and P.J.Eccles [2], [3], [4] studied the infinite loop spaces with the Γ^+-structure. While, D.G.Quillen defined the plus-construction of a space, furthermore J.B.Wagoner[12], A.T.Berrick[5], [6] and others [8] studied the plus-construction.

Thus, in this paper we shall study the infinite loop space with the plus-construction. Throughout this paper, we shall work in the category of based connected **CW**-complexes, which is denoted by $\Im op$. And all maps will mean the base point preserving maps unless otherwise stated. We denote the path space of a space X by $P(X)$, the maximal perfect normal subgroup of a group G by PG, and the loop functor by Ω.

2. Preliminaries

In this section, we consider the plus-construction of a space and its properties [5],[6],[12]. First we recall some Definitions.

DEFINITION 2.1. A fibration $F \longrightarrow E \longrightarrow B$ is **quasi-nilpotent** if the fundamental group $\pi_1(B)$ acts nilpotently on $H_*(F)$.

DEFINITION 2.2. A fibration $F \longrightarrow E \longrightarrow B$ is **nilpotent** provided that

(i) its fiber F is connected, and

(ii) the action of $\pi_1(E)$ on each $\pi_*(F)$ is nilpotent.

A space X is called a **nilpotent space** if the action of $\pi_1(X)$ on each $\pi_*(X)$ is nilpotent.

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DEFINITION 2.3. A space X is **acyclic** if $\widetilde{H}_*(X) = 0$, and a map $f: X \to Y$ is **acyclic** if the homotopy fiber $F_f = X \times_f P(Y)$ is an acyclic space.

Next we describe the acyclic fiber AX of a space X. Take $X_1 = X$, X_2 as a covering space of X with $\pi_1(X_2) = P\pi(X)$, and $X_3, X_4, \ldots$ are constructed by the pull-back as follows; there is a sequence of spaces

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_3 \to X_2$$

such that
(i) $\widetilde{H}_q(X_n) = 0 \qquad (q < n)$,
(ii) $X_{n+1} \longrightarrow X_n$ included form the path fibration $(n \geq 2)$ i.e.

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & P(K(H_n(X_n), n)) \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{\theta} & K(H_n(X_n), n)) \end{array} \quad \begin{array}{c} : cartesan \quad square. \\ (pull - back) \end{array}$$

(iii) X_n is unique up to fiber homotopy equivalence over X_{n-1}.

Here $K(H_n(X_n), n)$ is the Eilenberg-MacLane space. By AX, we denote the inverse limit space $\varinjlim X_n$. Then AX is an acyclic space and $AX \longrightarrow X_2$ is a nilpotent fiberation [6]. Usually X_n is called **the n-th term** of the acyclic tower of X .

Let X be a space.Then the space X^+, **the plus-construction** of X, is constructed as follows[6]; let $p: X' \longrightarrow X$ be a covering space of X with $\pi_1(X') = P\pi_1(X)$. Attach 2-cells and 3-cells to X' to get a simply connected space Y' such that $X' \xrightarrow{f'} Y'$ is an acyclic cofibration. Then we have the acyclic cofibration $X \xrightarrow{f} X^+$ by the push-out

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow[f]{} & X^+ \end{array}$$

with $Ker\pi_1(f) \quad = \quad P\pi_1(X)$. Usually the above map f is denoted by q_X .

Axiomatically we can characterize X^+ by the following statements[6]:

(a) $(q_X)_*: \pi_1(X) \longrightarrow \pi_1(X^+)$ is an epimorphism with kernel $P\pi_1(X)$

(b) The action of $\pi_1(X^+)$ on $\pi_n(X^+)$ is trival for each n.

(c) $(q_X)_*: H_*(X; Z) \longrightarrow H_*(X^+; Z)$ is an isomorphism.

(d) An acyclic map $f: X \longrightarrow Y$ is equivalent to the map q_X if and only if $P\pi_1(Y)$ is the trivial group.

(e) $(\Omega X)^+ = \Omega X$.

By above properties (a) and (d), we can define a map $f^+: X^+ \longrightarrow Y^+$ such that $f^+ \circ q_X = q_Y \circ f$, for any given map $f: X \longrightarrow Y$.

3. Main Theorems

In this section we shall construct the concrete infinite loop space by means of the delooping machine of the plus-construction. We denote the category of commutative rings with unity by $\mathfrak{R}$. First we recall some Lemmas in [6].

LEMMA 3.1. *Suppose that T and U are spaces and O is an acyclic space such that $T_n \to O \xrightarrow{p_{n+1}} U_{n+1}$ is a fiber sequence for some $n \geq 1$. Then the following three conditions are equivalent (see the following diagram):*

(a) p_{n+1} is quasi-nilpotent (and T_n^+ is nilpotent if n=1)

(b) $q_n: T_n \longrightarrow \Omega U_{n+1}^+$ is acyclic.

(c) $p_\infty: O \longrightarrow AU$ is acyclic.

$$\begin{array}{ccccc} T_n & \longrightarrow & O & \xrightarrow{p_{n+1}} & U_{n+1} \\ {\scriptstyle q_{n+1}}\downarrow & & {\scriptstyle p_\infty}\downarrow & & \| \\ \Omega U_{n+1}^+ & \longrightarrow & AU & \longrightarrow & U_{n+1} \end{array}$$

LEMMA 3.2. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. If p is quasi-nilpotent and F^+ is nilpotent, then $F^+ \xrightarrow{i^+} E^+ \xrightarrow{p^+} B^+$ is also a fibration.*

LEMMA 3.3. *Suppose $f: X \longrightarrow Y$ is a quasi-nilpotent fibration. Then X is nilpotent if and only if both the homotopy fiber F_f and Y are nilpotent.*

From the above Lemmas, we have

THEOREM 3.4. *Let $T \longrightarrow O \xrightarrow{p_2} U_2$ be a quasi-nilpotent fibration such that O is acyclic and U_2 is the second term of the acyclic tower of U. If T^+ is nilpotent then $q_n^+: T_n^+ \longrightarrow \Omega U_{n+1}^+$ is a homotopy equivalence for all $n \geq 1$.*

Proof. Since p_{n+1} is a quasi-nilpotent fibration and $T^+ = T_1^+$ is nilpotent, $q_n: T_n \longrightarrow \Omega U_{n+1}^+$ is acyclic by Lemma 3.1. Thus q_n is a quasi-nilpotent homotopy equivalent map. Therefore we may consider the fiber sequence $F \longrightarrow T_n \xrightarrow{q_n} \Omega U_{n+1}^+$ such that $\widetilde{H}_*(F) = \widetilde{H}_*(F^+) = 0$. Since $\pi_1(T_n^+)$ is the trivial group $(n > 1)$ and the action $\pi_1(T_n^+) \times \pi_*(T_n^+) \longrightarrow \pi_*(T_n^+)$ is nilpotent, T_n^+ is a nilpotent space. Furthermore, F^+ is also a nilpotent space, by Lemma 3.2, we can make another fiber sequence $F^+ \longrightarrow T_n^+ \longrightarrow (\Omega U_{n+1}^+)^+ = \Omega U_{n+1}^+$ where q_n^+ is also acyclic. Thus q_n^+ is a quasi-nilpotent homotopy equivalent map. Moreover, by Lemma 3.3, ΩU_{n+1}^+ is a nilpotent space and the above map $q_n^+: T_n^+ \longrightarrow \Omega U_{n+1}^+$ is a nilpotent map. Hence q_n^+ is a nilpotent homotopy equivalence. Thus $q_n^+: T_n^+ \longrightarrow \Omega U_{n+1}^+$ is a homotopy equivalence.

COROLLARY 3.5. *If $S: \mathfrak{R} \longrightarrow \mathfrak{R}$ and $T: \mathfrak{R} \longrightarrow \mathfrak{Top}$ are covariant functors such that for any object A in $\mathfrak{R}$ there is an acyclic space O_A in $\mathfrak{Top}$ and a quasi-nilpotent fibration $T(A) \longrightarrow O_A \longrightarrow T(SA)_2$ with $T(A)^+$ nilpotent. Then there exists an Ω-spectrum*

$$\cdots, \Omega^2 T(A)^+, \quad \Omega T(A)^+, \quad T(A)^+, \quad T(SA)_2^+, \quad T(S^2A)_3^+, \cdots.$$

That is, $T(A^+)$ is an infinite loop space.

Proof. From the above Theorem 3.4, put $U_2 = T(SA)_2, O = O_A$ and $T = T(A)$. Then there exists a homotopy equivalence $T(A)_n^+ \longrightarrow \Omega T(SA)_{n+1}^+$. Therefore we can make the following Ω-spectrum;

$$\cdots, \Omega^2 T(A)^+, \quad \Omega T(A)+, \quad T(A)^+, \quad T(SA)_2^+, \quad T(S^2A)_3^+, \cdots.$$

Finally we shall prove the following.

THEOREM 3.6. *$BGLA$ is an infinite loop space, where A is an object in $\mathfrak{R}$ and B the classifying space functor.*

In order to prove the above Theorem 3.5, we need some notations and a Lemma in [6]. Let CA be the ring of locally finite matrices over A, and

$MA(\subset CA)$ be the two-sided ideal of finite matrices, i.e. those matrices have at most finitely many non-zero entries. Define $SA = CA/MA$ which is called the *suspension ring* of A. Obviously S is a functor from $\Re$ to itself.

LEMMA 3.7. *T^+ is nilpotent if and only if $\pi_1(T)/P\pi_1(T)$ is nilpotent and the fibration $T_2 \longrightarrow T \longrightarrow K(\pi_1(T)/P\pi_1(T), 1)$ is quasi-nilpotent.*

Proof of Theorem 3.6. Since the fibration

$$BGLA \longrightarrow BGLCA \longrightarrow (BGLSA)_2 = BESA$$

is quasi-nilpotent and $BGLCA$ is an acyclic space, it suffices to prove that $BGLA^+$ is a nilpotent space, because of corollary 3.5. Consider the quasi-nilpotent fibration

$$BEA = (BGLA)_2 \longrightarrow BGLA \longrightarrow K(\pi_1(BGLA)/P\pi_1(BGLA), 1).$$

Then we have that

$$\pi_1(BGLA)/P\pi_1(BGLA) = GLA/EA$$

is a nilpotent group. Therefore, by Lemma 3.7, $BGLA^+$ is a nilpotent space.

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