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# AN EXACT SEQUENCE IN THE UNITARY EQUIVALENT COBORDISM THEORY

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The cobordism theory was introduced by *R.* Thom in his earlier paper "Quelques propriètès globales des variètès differentiables, Comment. Math. Helv. 28 (1954) pp. 17–8". Later P.E. Conner and E.E. Floyd applied the cobordism theory to study differentiable manifolds with structure groups in four papers which were published during 1964–1966. The equivalent cobordism theory is a result of P.E. Conner and E.E. Floyd. Moreover, the bordism theory and the cobordism theory are deeply related to each other in spite of the difference of their definitions ([1], [2], [4]). It is also well-known that the cobordism theory is applied to the index theory of Atiyah-Singer.

The purpose of this paper is to prove an exact sequence which is induced from the unitary equivalent cobordism theory and a new idea in Definition 3 (Theorem 5).

Throughout the paper we assume that $G$ is a compact Lie group. Let $\mathcal{D}(G)$ be the category consisting of all topological $G$-spaces with base points and all base point preserving continuous $G$-maps ([5]). For $X, Y \in \mathrm{Obj}(\mathcal{D}(G))$ by $[X,Y]_0^G$ we mean the set of all base point preserving $G$ homotopy classes, where $f_0, f_1 : X \longrightarrow Y$ in $\mathrm{Morph}(\mathcal{D}(G))$ are $G$-homotopic if there exists a homotopy $f_t : X \longrightarrow Y$ in $\mathrm{Morph}(\mathcal{D}(G))$ for each $t \in [0,1] = I$. In particular, $I$ is in $\mathrm{Obj}(\mathcal{D}(G))$ with the base point 0 and the trivial $G$-action ($\forall\, t \in I$and $\forall\, g \in G, g \cdot t = t$).

For a $G$-space $X \in \mathrm{Obj}(\mathcal{D}(G))$ with its base point $x_0$ we put

$$CX = X \times I/(x_0 \times I \cup X \times 0)$$

which is called the cone of $X$ and

$$SX = X \times I/(x_0 \times I \cup X \times \{0,1\})$$

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which is called the suspension of $X$. For a morphism $f : X \longrightarrow Y$ in Morph($\mathcal{D}(G)$) we define

$$C_f = X \times I \cup Y / \sim$$

where $(x,1) \sim f(x)$, $(x_0,t) \sim y_0$, $(x,0) \sim y_0$ for $x \in X$, the base point $x_0$ of $X$ and the base point $y_0$ of $Y$. Then there exist the inclusion

$$a(f) : Y \longrightarrow C_f$$

and the projection

$$b(f) : C_f \longrightarrow C_f / a(f)(Y) = SX.$$

The constant map $c : X \longrightarrow Y$ in Morph($\mathcal{D}(G)$) is denoted by $[c] = 0$ in $[X,Y]_0^G$. The $[X,Y]_0^G$is in Obj($\mathcal{D}(G)$), where the topology of $[X,Y]_0^G$ is the compact-open topology and 0 is the base point of $[X,Y]_0^G$.

For each morphism $f : X \longrightarrow Y$ in Morph($\mathcal{D}(G)$), the Barratt-Puppe sequence is

$$(\alpha) : X \xrightarrow{f} Y \xrightarrow{a(f)} C_f \xrightarrow{b(f)} SX \xrightarrow{Sf} SY \longrightarrow \cdots$$

and for each $W \in$ Obj($\mathcal{D}(G)$) the sequence

$$[(\alpha) : W]_0^G : [X,W]_0^G \longleftarrow [Y,W]_0^G \longleftarrow [C_f,W]_0^G \longleftarrow [SX,W]_0^G \longleftarrow \cdots$$

is exact ([3], [5]).

For each $f : X \longrightarrow Y \in$ Morph($\mathcal{D}(G)$) we define

$$E_f = \{(x,\omega) \in X \times PY | f(x) = \omega(1)\},$$

where $PY = \{\omega : I \longrightarrow Y | \omega(0) = y_0$ (base point of $Y$) and $\omega$ is continuous$\}$ (note that $\omega$ is not necessarily a $G$-map). Then $E_f$ is a $G$-space with product topology and $(x_0, 0)$ as the base point where $x_0$ is the base point of $X$ and $0(t) = y_0$ for all $t \in I$. The action of $G$ on $E_f$ is $g(x,\omega) = (gx, g\omega)$ where $g \in G, (x,\omega) \in E_f$ and $g\omega = \{g\omega(t) | t \in [0,1]\}$. Then, it is well-known that

$$E_f \xrightarrow{j_f} X \xrightarrow{f} Y$$

is a fibration where $j_f : E_f \longrightarrow X$ is defined by $j_f((x,\omega)) = x$ which is a $G$-map.

LEMMA 1. *Let $X \in \mathrm{Obj}(\mathcal{D}(G))$ be (n-1)-connected and $f : X \longrightarrow Y$ be in* Morph($\mathcal{D}(G)$). *If $C_f(\in \mathrm{Obj}(\mathcal{D}(G)))$ is (m-1)-connected and $W(\in \mathrm{Obj}(\mathcal{D})(G)))$ is a connected CW-complex with dim $W = r \leq n + m - 2$ then*

$$[W, X]_0^G \xrightarrow{f_*} [W, Y]_0^G \xrightarrow{a(f)_*} [W, C_f]_0^G$$

*is exact. Moreover, if $Y \in \mathrm{Obj}(\mathcal{D}(G))$ is $(l-1)$-connected with* dim $W = r \leq l + n - 1$ *then*

$$[W, X]_0^G \xrightarrow{f_*} [W, Y]_0^G \xrightarrow{a(f)_*} [W, C_f]_0^G \xrightarrow{b(f)_*} [W, SX]_0^G \longrightarrow \cdots$$

*is exact ([3]).*

*Proof.* We shall sketch this proof as follows. Since

$$E_{a(f)} \xrightarrow{j_a(f)} Y \xrightarrow{a(f)} C_f$$

is a fibration we have the exact sequence

$$[W, E_{a(f)}]_0^G \longrightarrow [W, Y]_0^G \longrightarrow [W, C_f]_0^G$$

([3]). Define

$$\rho : X \longrightarrow E_{a(f)} \text{ by } \rho(x) = (f(x), \omega_x),$$

where $\omega_x : I \longrightarrow C_f$ is defined by $\omega_x(t) = (1-t, f(x)) \in C_f$ for $t \in [0,1]$. Then, in our case, $\rho : X \longrightarrow E_{a(f)}$ is (m+n-2)-connected, i.e.,

$$H_i(\rho) : H_i(X) \longrightarrow H_i(E_{a(f)}) \text{ and } \pi_i(\rho) : \pi_i(X) \longrightarrow \pi_i(E_{a(f)}),$$

where

$$\begin{array}{ll} H_i(\rho) \text{ and } \pi_i(\rho) \text{ are isomorphisms} & \text{if } i < m+n-2 \\ H_i(\rho) \text{ and } \pi_i(\rho) \text{ are surjective} & \text{if } i \leq m+n-2. \end{array}$$

Hence $\rho_* : [W, X]_0^G \longrightarrow [W, E_{a(f)}]_0^G$ is surjective and thus

$$[W, X]_0^G \longrightarrow [W, Y]_0^G \longrightarrow [W, C_f]_0^G$$

is exact. Similarly, for the cofibration

$$Y \xrightarrow{a(f)} C_f \xrightarrow{b(f)} SX$$

we have the exact sequence

$$[W,Y]_0^G \xrightarrow{a(f)_*} [W,C_f]_0^G \xrightarrow{(Sf)_*} [W,SX]_0^G$$

because that $Y$ is ($l$-1)-connected, $SX$ is $n$-connected and dim $W = r \leq n+l-1$. Repeating this way we have the exact sequence

$$[W,X]_0^G \longrightarrow [W,Y]_0^G \longrightarrow [W,C_f]_0^G \longrightarrow [W,SX]_0^G \longrightarrow \cdots .$$

For any $X$ and $Y$ in Obj($\mathcal{D}(G)$) the wedge product of $X$ and $Y$ is defined by

$$X \vee Y = \{(x,y_0)|x \in X\} \cup \{(x_0,y)|y \in Y\}$$

where $x_0$ is the base point of $X$ and $y_0$ the base point of $Y$, and the smash product of $X$ and $Y$ is

$$X \wedge Y = X \times Y/X \vee Y.$$

It is clear that if for $X, Y_1, Y_2$ and $Y_3$ in Obj($\mathcal{D}(G)$)

$$[X,Y_1]_0^G \longrightarrow [X,Y_2]_0^G \longrightarrow [X,Y_3]_0^G$$

is exact then for any $Z \in$ Obj($\mathcal{D}(G)$)

$$(*) \qquad [X,Y_1 \wedge Z]_0^G \longrightarrow [X,Y_2 \wedge Z]_0^G \longrightarrow [X,Y_3 \wedge Z]_0^G$$

is also exct.

Let $V(k)$ be a $k$-dimensional complex $G$-vector space, and let $B_n(V(k))$ be the Grassmann manifold consisting of all $n$-dimensional subspaces of $V(k)(n \leq k)$. For each $W \in B_n(V(k))$ and $g \in G$ we define

$$g \cdot W = \{gv|v \in W\}$$

then $B_n(V(k))$ is a differentiable $G$-space ([5]). We put

$$E_n(V(k)) = \{(W, w) | W \in B_n(V(k)), w \in W\}$$

and define

$$g(W, w) = (gW, gw).$$

Then it is clear that

$$\pi : E_n(V(k)) \longrightarrow B_n(V(k)) \text{ by } \pi(W, w) = W$$

is a differentiable $G$-vector bundle with dimension $n$. We define the one-point compactification

$$E_n(V(k)) \cup \{\infty\} = M_n(V(k))(\in \mathrm{Obj}(\mathcal{D}(G)))$$

which is called the Thom space of $\pi : E_n(V(k)) \longrightarrow B_n(V(k))$. Let

$$i(k, l) : V(k) \longrightarrow V(l) \quad (k \leq l)$$

be the inclusion, then $\{E_n(V(k)), i(k, l)_*\}$ is an inductive limit system. We put

$$E_n(G) = \varinjlim_k E_n(V(k)).$$

Let

$$M(i(k, l)_*) : M_n(V(k)) \longrightarrow M_n(V(k)) \quad (k \leq l)$$

be induced form

$$i(k, l)_* : E_n(V(k)) \longrightarrow E_n(V(l)).$$

Then $\{M_n(V(k)), M(i(k, l)_*)\}$ is an inductive limit system, and thus we put

$$M_n(G) = \varinjlim_k M_n(V(l)).$$

We also define the $G$-isomorphism

$$s(k) : V(k) \oplus V(k) \longrightarrow V(2k)$$

by

$$s(k)((v_1,\cdots,v_k)\oplus(w_1,\cdots,w_k))=(v_1,\cdots,v_k,w_1,\cdots,w_k).$$

The injective linear $G$-map

$$s(k,l):V(k)\oplus V(l)\longrightarrow V(2k+2l)$$

is defined by the composition $s(k+l)\circ(i(k,k+l)\oplus i(l,k+l))$.

For the $G$-vector bundle morphism

$$a(V(k),V(l):E_m(V(k))\times E_n(V(l))\longrightarrow E_{m+n}(V(k)\oplus V(l))$$

by

$$a(V(k),V(l))((W_1,w_1)\times(W_2,w_2))=(W_1\oplus W_2,w_1\oplus w_2)$$

we define the $G$-vector bundle morphism

$$a(k,l):E_m(V(k))\times E_n(V(l))\longrightarrow E_{m+n}(V(2k+2l))$$

by the composition $s(k,l)_*\circ a(V(k),V(l))$, where $s(k,l)_*$ is induced from $s(k,l)$. Moreover, the $G$-vector bundle morphism $a(k,l)$ induces the $G$-vector bundle morphism

$$a_{m,n}:E_m(G)\times E_n(G)\longrightarrow E_{m+n}(G)$$

and $a_{m,n}$ induces the $G$-map

$$M(a_{m,n}):M_m(G)\wedge M_n(G)\longrightarrow M_{m,n}(G).$$

It is obvious that there exists the inclusion

$$i(k):V(k)^c=V(k)\cup\{\infty\}\quad(\cong S^{2k})\to M_k(G)$$

where $k$ is a non negative integer. The map

$$\varepsilon=\varepsilon_{k,n}:V(k)^c\wedge M_n(G)\longrightarrow M_{n+k}(G)$$

is defined by the commutative diagram :

$$\begin{array}{ccc} V(k)^C\wedge M_n(G) & \xrightarrow{\varepsilon_{k,n}} & M_{n+k}(G) \\ & {\scriptstyle i(k)\wedge 1}\searrow \quad \swarrow{\scriptstyle M(a_{k,n})} & \\ & M_k(G)\wedge M_n(G) & \end{array}$$

We also define the $G$-isomorphism

$$e=e(k,l):V(k)\oplus V(l)\longrightarrow V(k+l)$$

by

$$e(k,l)((v_1,\cdots,v_k)\oplus(w_1,\cdots,w_l))=(v_1,\cdots v_k,w_1,\cdots,w_k).$$

DEFINITION 2. For $X, Y \in \mathrm{Obj}(\mathcal{D}(G))$ and for non negative integers $k$ and $n$ we put

$$U_G^{2n}(k)(X:Y) = |V(k)^c \wedge X, M_{k+n}(G) \wedge Y|_0^G,$$

and for a non negative integer $l$ we define

$$e(l) : U_G^{2n}(k)(X:Y) \longrightarrow U_G^{2n}(k+l)(X:Y)$$

by the homotopy class of composition

$$V(k+l)^c \wedge X \xrightarrow{e^{-1}\wedge 1} V(l)^c \wedge V(k)^c \wedge X \xrightarrow{l\wedge f} V(l)^c \wedge M_{n+k}(G) \wedge Y$$
$$\xrightarrow{\varepsilon\wedge 1} M_{n+k+l}(G) \wedge Y$$

where $[f] \in U_G^{2n}(k)(X:Y)$. Then $\{U_G^{2n}(k)(X:Y), e(l)\}$ is an inductive limit system ([5]). Then

$$\tilde{U}_G^{2n}(X:Y) = \varinjlim_k U_G^{2n}(k)(X:Y)$$

is an abelian group ([5]) which is called the $2n$-dimensional unitary equivalent cobordism group of $X$ and $Y$ with action group $G$. The unitary equivalent cobordism theory is defined from the exact sequence $[(\alpha) : W]$

We define the natural transformation

$$\sigma_*(l) : U_G^{2n}(k)(X : V(l)^c \wedge Y) \longrightarrow U_G^{2n+2l}(k)(X:Y)$$

by the homotopy class of the composition

$$V(k)^c \wedge X \xrightarrow{f} M_{n+k}(G) \wedge V(l)^c \wedge Y \xrightarrow{T\wedge 1} V(l)^c \wedge M_{n+k}(G) \wedge Y$$
$$\xrightarrow{\varepsilon\wedge 1} M_{n+k+l}(G) \wedge Y$$

where $[f] \in U_G^{2n}(X : V(l)^C \wedge Y)$ and $T(x \wedge y) = y \wedge x$ ([5]). It is clear that the diagram

$$\begin{array}{ccc} U_G^{2n}(X : V(l)^C \wedge Y) & \xrightarrow{\sigma_*(l)} & U_G^{2n+2l}(k)(X:Y) \\ {\scriptstyle e(k')}\downarrow & & \downarrow{\scriptstyle e(k')} \\ U_G^{2n}(k+k')(X : V(l)^C \wedge Y) & \xrightarrow{\sigma_*(l)} & U_G^{2n+2l}(k+k')(X:Y) \end{array}$$

is commutative, and which induces the $G$-isomorphism

$$\sigma_*(l) : \tilde{U}_G^{2n}(X : V(l)^c \wedge Y) \xrightarrow{\cong} \tilde{U}_G^{2n+2l}(X : Y)$$

([5]).

DEFINITION 3. For $n = 0, 1, 2, \cdots$ we define

$$\tilde{U}_G^{2n+1}(X : Y) = \tilde{U}_G^{2n}(X : SY),$$

where $X, Y \in \mathrm{Obj}(\mathcal{D}(G))$. We put

$$\sigma'_* = \text{the identity } : \tilde{U}_G^{2n+1}(X : Y) \longrightarrow \tilde{U}_G^{2n}(X : SY).$$

The $G$-isomorphism

$$\tilde{\sigma}_* : \tilde{U}_G^{2n}(X : Y) \xrightarrow{\cong} \tilde{U}_G^{2n-1}(X : SY)$$

is defined by the commutative diagram ($S^2 \cong \mathcal{C}^c$) :

$$\begin{array}{ccc} \tilde{U}_G^{2n}(X : Y) & \xrightarrow{\tilde{\sigma}_*} & \tilde{U}_G^{2n-1}(X : SY) \\ {\scriptstyle \sigma_*(1)^{-1}}\big\downarrow & & \cong\big\downarrow{\scriptstyle \sigma'^{-1}_*} \\ \tilde{U}_G^{2n-2}(X : S^2Y) & = & \tilde{U}_G^{2n-2}(X : S(SY)). \end{array}$$

In consequence we have defined the $G$-isomorphism

$$\sigma_* : \tilde{U}_G^{n+1}(X : Y) \longrightarrow \tilde{U}_G^{n}(X : SY).$$

LEMMA 4. *For each $f : Y \longrightarrow Y'$ in* Morph($\mathcal{D}(G)$) *the diagram*

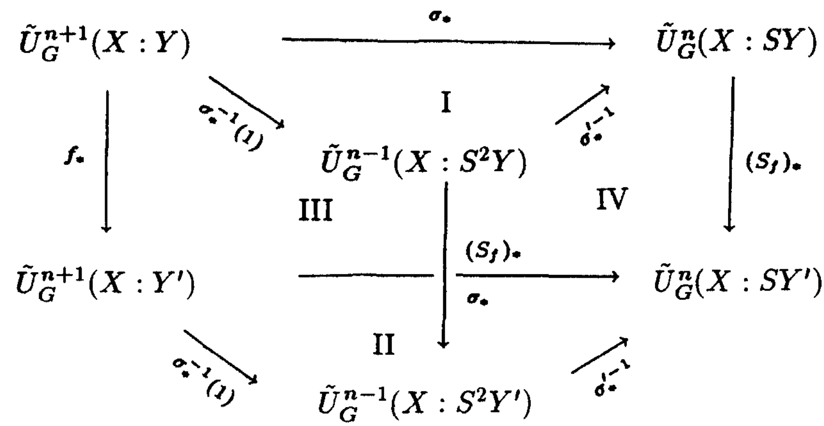
$$\begin{array}{ccc} \tilde{U}_G^{n+1}(X : Y) & \xrightarrow{\sigma_*} & \tilde{U}_G^{n}(X : SY) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (Sf)_*} \\ \tilde{U}_G^{n+1}(X : Y') & \xrightarrow{\sigma_*} & \tilde{U}_G^{n}(X : SY'). \end{array}$$

*is commutative, where* $X \in \mathrm{Obj}(\mathcal{D}(G))$.

*Proof.* In the following diagram,

(i) triangles I and II are commutative by Definition 3

(ii) squares III and IV are commutative since $\sigma_*(1)^{-1}$ and $\sigma'_*$ are natural maps :




Therefore $(S_f)_* \circ \sigma_* = \sigma_* \circ f_*$.

THEOREM 5. *Let* $Y \in \mathrm{Obj}(\mathcal{D}(G))$ *be* $\omega$*-connected. Then, for* $f : Y \longrightarrow Y'$ *in* $\mathrm{Morph}(\mathcal{D}(G))$ *(*$Y'$*:connected) and a finite dimensional connected CW complex* $X$ *in* $\mathrm{Obj}(\mathcal{D}(G))$*, we have the exact sequence*

$$\begin{aligned} \cdots \longrightarrow \tilde{U}_G^{n}(X:Y) \longrightarrow \tilde{U}_G^{n}(X:Y') \longrightarrow \tilde{U}_G^{n}(X:C_f) \\ \longrightarrow \tilde{U}_G^{n+1}(X:Y) \longrightarrow \tilde{U}_G^{n+1}(X:Y') \longrightarrow \tilde{U}_G^{n+1}(X:C_f) \\ \longrightarrow \tilde{U}_G^{n+2}(X:Y) \longrightarrow \cdots \end{aligned}$$

*Proof.* By Lemma 1 (note that $\dim S^kX \leq \infty$) for some non-negative integers $k, k'$ and $n$ following diagram is commutative :

$$\begin{array}{ccccccc} U_G^{2n}(k)(X:Y) & \longrightarrow & U_G^{2n}(k)(X:Y') & \longrightarrow & U_G^{2n}(k)(X:C_f) & : \text{exact} \\ \downarrow e(k') & © & \downarrow e(k') & © & \downarrow e(k') & \\ U_G^{2n}(k+k')(X:Y) & \longrightarrow & U_G^{2n}(k+k')(X:Y') & \longrightarrow & U_G^{2n}(k+k')(X:C_f) & : \text{exact} \end{array}$$

(see(*) above). Thus the sequence

$$(**) \qquad \tilde{U}_G^{2n}(X:Y) \longrightarrow \tilde{U}_G^{2n}(X:Y') \longrightarrow \tilde{U}_G^{2n}(X:C_f)$$

is exact. Define

$$\delta : \tilde{U}_G^n(X:C_f) \longrightarrow \tilde{U}_G^{n+1}(X:Y)$$

by the composition

$$\tilde{U}_G^n(X:C_f) \xrightarrow{b(f)_*} \tilde{U}_G^n(X:SY) \xrightarrow{\cong \sigma_*^{-1}} \tilde{U}_G^{n+1}(X:Y).$$

Then, by Lemma 4 and the above exact sequence (**) it is proved that the sequence

$$\cdots \xrightarrow{\delta} \tilde{U}_G^n(X:Y) \to \tilde{U}_G^n(X:Y') \to \tilde{U}_G^n(X:C_f) \xrightarrow{\delta} \tilde{U}_G^{n+1}(X:Y) \to \cdots$$

is exact.

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