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ISÉKI'S CONDITION (S) ON SOME BCK–ALGEBRAS

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1. Introduction and preliminaries

In 1966, Iséki[3] and Imai and Iséki[2] introduced the notion of BCK-algebras. A BCK-algebra is a nonempty set X together with a binary operation $*$ and a special element 0 for which the following axioms hold for all $x, y, z \in X$:

(1) $(x * y) * (x * z) \leq z * y$,
(2) $x * (x * y) \leq y$,
(3) $x \leq x$,
(4) $0 \leq x$,
(5) $x \leq y$ and $y \leq x$ imply $x = y$,

where $x \leq y$ is defined by $x * y = 0$.

In any BCK-algebra X, the following relations hold for all $x, y, z \in X$:

(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
(7) $x \leq y$ and $y \leq z$ imply $x \leq z$.
(8) $(x * y) * z = (x * z) * y$.
(9) $x * y \leq x$.
(10) $x * 0 = x$.

A BCK-algebra X is said to be *bounded* if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. If $x \wedge y = y \wedge x$ for all $x, y \in X$, where $x \wedge y = y * (y * x)$, then X is said to be *commutative*. We say that X is *positive implicative* if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$, and that X is *implicative* if $x * (y * x) = x$ for all $x, y \in X$. Every implicative BCK-algebra is commutative and positive implicative(see [5]).

A nonempty subset I of a BCK-algebra X is called an *ideal* of X if

(11) $0 \in I$,
(12) $x \in I$ and $y * x \in I$ imply that $y \in I$.

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In [4], Iséki introduced the notion of a BCK-algebra with condition (S). The object of this paper is to investigate the Iséki's condition (S) on some BCK-algebras.

2. Condition (S) on some BCK-algebras

DEFINITION 2.1([4]). A BCK-algebra X is said to be with condition (S) if for any fixed elements $y, z \in X$, the set

$$A(y,z) = \{x \in X : x * y \leq z\}$$

has the greatest element which we denote by $y \circ z$.

In any BCK-algebra X with condition (S), the following hold for all $x, y, z \in X$ (see [4]):
(13) $x \circ 0 = 0 \circ x = x$.
(14) $x * (y \circ z) = (x * y) * z$.
In case X is also implicative, then
(15) $(x \circ y) * z = (x * z) \circ (y * z)$.
(16) $x \circ x = x$.

DEFINITION 2.2([1]). Let X be a commutative BCK-algebra and Y a subalgebra of X. X will be called a quotient BCK-algebra of Y if for every $x_1, x_2, x \in X$ with $x_1 \neq x_2$, there exists an element $y \in Y$ such that $x_1 \wedge y \neq x_2 \wedge y$ and $x \wedge y \in Y$.

DEFINITION 2.3([1]). An ideal I of a bounded implicative BCK-algebra X is said to be dense in X if X is a quotient BCK-algebra of I.

DEFINITION 2.4([1]). Let I be an ideal of a bounded implicative BCK-algebra X. A mapping $f : I \to X$ is called a partial $\wedge$-homomorphism if $f(x \wedge y) = f(x) \wedge y$ for all $x \in I$ and $y \in X$.

We denote the set of all partial $\wedge$-homomorphisms from an ideal I of a bounded implicative BCK-algebra X into X by $Hom_X(I, X)$. It is shown in [1] that $Hom_X(I,X)$ is a bounded implicative BCK-algebra if I is a dense ideal of X.

PROPOSITION 2.5. *Let X be a bounded implicative BCK-algebra with condition (S) and I a dense ideal of X. Then the algebra $Hom_X(I,X)$ satisfies also condition (S).*

Proof. Define an operation $\circ$ on $Hom_X(I,X)$ by

$$(f \circ g)(x) = f(x) \circ g(x)$$

for all $x \in I$ and $f, g \in Hom_X(I,X)$. Then $f \circ g$ is clearly well-defined. Now,

$$\begin{aligned}
&\{(f \circ g) * f\}(x) \\
=&(f \circ g)(x) * f(x) \\
=&\{f(x) \circ g(x)\} * f(x) \\
=&\{f(x) * f(x)\} \circ \{g(x) * f(x)\} && \text{by } (15) \\
=&0 \circ \{g(x) * f(x)\} \\
=&g(x) * f(x) && \text{by } (13) \\
\leq&g(x) && \text{by } (9)
\end{aligned}$$

for all $x \in I$. This shows that $(f \circ g) * f \leq g$. Let $h \in Hom_X(I,X)$ be such that $h * f \leq g$. Then

$$\begin{aligned}
&\{h * (f \circ g)\}(x) \\
=&h(x) * \{f(x) \circ g(x)\} \\
=&\{h(x) * f(x)\} * g(x) && \text{by } (14) \\
=&(h * f)(x) * g(x) \\
=&0
\end{aligned}$$

for all $x \in I$, which implies that $h * (f \circ g) = 0$, that is, $h \leq f \circ g$. This proves that $Hom_X(I,X)$ satisfies condition (S).

Let X be a bounded implicative BCK-algebra and we set

$$H_X = \cup\{Hom_X(I,X) : I \in X^{\Delta}\}$$

where X^{Δ} is the set of all dense ideals of X. Define a binary operation $*$ and an order relation $\leq$ on H_X as follows: for $f, g \in H_X$,

$$(f * g)(x) = f(x) * g(x)$$

and

$$f \leq g \Longleftrightarrow f(x) \leq g(x)$$

for all $x \in I_f \cap I_g$, where I_f and I_g are the domain of f and g respectively. It is shown in [1] that H_X is a bounded implicative BCK-algebra.

PROPOSITION 2.6. *If X is a bounded implicative BCK-algebra with condition (S), then H_X is also with condition (S).*

Proof. Let $f, g \in H_X$. Then $f \in Hom_X(I_f, X)$ and $g \in Hom_X(I_g, X)$ for some $I_f, I_g \in X^{\Delta}$. An operation $\circ$ on H_X defined by $(f \circ g)(x) = f(x) \circ g(x)$ for all $x \in I_f \cap I_g$ is well-defined, and

$$\begin{aligned}
&\{(f \circ g) * f\}(x) \\
=&\{f(x) \circ g(x)\} * f(x) \\
=&\{f(x) * f(x)\} \circ \{g(x) * f(x)\} \\
=&g(x) * f(x) \\
\leq&g(x)
\end{aligned}$$

for all $x \in I_f \cap I_g$. This implies that $(f \circ g) * f \leq g$. Let $h \in H_X$ be such that $h * f \leq g$. Then $h \in Hom_X(I_h, X)$, $I_h \in X^{\Delta}$ and $h * f \leq g$. We now see that

$$\begin{aligned}
&\{h * (f \circ g)\}(x) \\
=&h(x) * \{f(x) \circ g(x)\} \\
=&\{h(x) * f(x)\} * g(x) \\
=&(h * f)(x) * g(x) \\
=&0
\end{aligned}$$

for all $x \in I_f \cap I_g \cap I_h$. This means that $h \leq f \circ g$, showing that H_X satisfies condition (S).

Define a relation $\sim$ on H_X as follows: for $f, g \in H_X$,

$$f \sim g \Longleftrightarrow f(x) = g(x) \text{ on some } I \subset I_f \cap I_g.$$

Clearly $\sim$ is an equivalence relation . For an element $f \in H_X$, its equivalence class will be denoted by $[f]$ and $Q(X)$ will denote the set of all equivalence classes. We now define a binary operation $*$ and an order relation $\leq$ on $Q(X)$ as follows: for $[f], [g] \in Q(X)$,

$$[f] * [g] = [f * g]$$

and

$$[f] \leq [g] \Longleftrightarrow f \leq g.$$

Then $Q(X)$ is a bounded implicative BCK-algebra(see [1]).

PROPOSITION 2.7. *Let X be a bounded implicative BCK-algebra with condition (S). Then $Q(X)$ satisfies also condition (S).*

Proof. Let $[f], [g] \in Q(X)$ and define an operation $\circ$ on $Q(X)$ as follows:

$$[f] \circ [g] = [f \circ g].$$

Suppose that $[f] = [f']$ and $[g] = [g']$, where $[f], [f'], [g], [g'] \in Q(X)$. Then $f \sim f'$ and $g \sim g'$, that is, $f(x) = f'(x)$ and $g(x) = g'(x)$ on some $I' \subset I_f \cap I_{f'}$ and $I" \subset I_g \cap I_{g'}$ respectively. It follows that $f(x) \circ g(x) = f'(x) \circ g'(x)$ for all $x \in I = I' \cap I"$. This means that $f \circ g \sim f' \circ g'$, that is , $[f \circ g] = [f' \circ g']$. Hence $[f] \circ [g]$ is well-defined. Now,

$$\begin{aligned} &\{[f] \circ [g]\} * [f] \\ =&[f \circ g] * [f] \\ =&[(f \circ g) * f] \\ \leq&[g]. \end{aligned}$$

Let $[h] \in Q(X)$ be such that $[h] * [f] \leq [g]$. Then

$$\begin{aligned} &[h] * ([f] \circ [g]) \\ =&[h * (f \circ g)] \\ =&[(h * f) * g] \\ =&([h] * [f]) * [g] \\ =&[0]. \end{aligned}$$

Hence $[h] \leq [f] \circ [g]$. This completes the proof.

By [4; Theorem 1 and Corollaries 6 - 7], we get the following:

COROLLARY 2.8. *Let X be a bounded implicative BCK-algebra and I a dense ideal of X. Then the algebras $Hom_X(I, X)$, H_X and $Q(X)$ are all commutative semigroups, semi-Brouwerian algebras and regular Griss algebras.*

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