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NON-ORIENTABLE MANIFOLDS WITH A TOTAL ACTION

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0. Introduction

In section 1, we are going to study two types of those 4-manifolds with a T^2 action which we encounter in a classifying problem. As a standard technique, by investigating the orbit spaces, we will get some informations of the total spaces.

J. Pak classified, in [P], $(n+1)$-orientable manifolds with a T^2 action. Since it is not found in any literature for the nonorientable case, we gave the complete solution in section 2.

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1. 4-Manifolds with a T^2 action

Througout the paper, we adopt the following notation. Let S^1 be the set of all complex numbers whose absolute value is 1, we denote $\exp 2\pi\imath\phi$ as a point of S^1, where ϕ is a real number and exp is the exponential function. Let S^2 be the unit sphere in the 3 - dimensional Euclean space R^3. We will use $(\rho \exp 2\pi\imath\theta, z)$ or ν as a point of S^2, where $0 \leq \rho \leq 1$, $-1 \leq z \leq 1$, since R^3 can be identified with the product of the complex plane $\mathbb{C}$ and R^1. Let T^n be the n times product of S^1. Every manifold is smooth and closed.

DEFINITION 1.1. Let f be a function from R^2 to the 2 - dimensional complex plance $\mathbb{C}^2$ defined by $f(x,y) = (\exp 2\pi\imath x, \exp 2\pi\imath y)$. Given relatively prime integers m, n, we define the image of the straight line $mx + ny = 0$ in R^2 under f to be (m,n).

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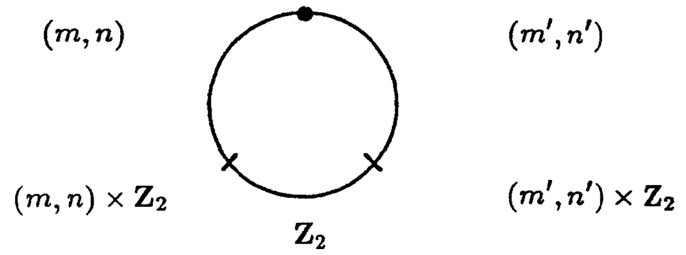
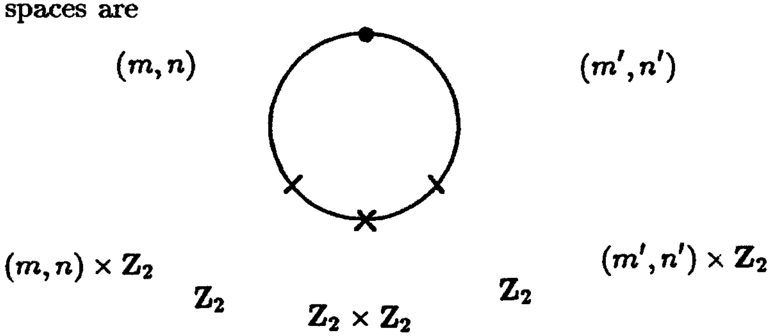
DEFINITION 1.2. A group G action on M is effective if $gx = x$, for all x in M, implies g is the identity element in G. We denote the quotient space (i.e. orbit space with "weights") by M/G. For example,

$M/T^2 =$

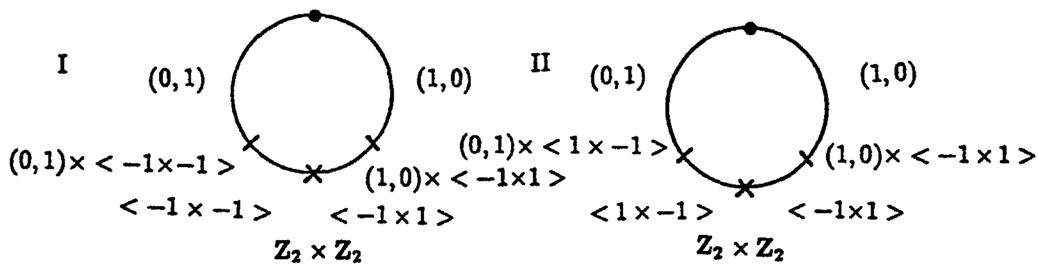


is a weighted disk. The weight at each interior point is the identity, while we have divided up the boundary into 3 arcs. The 3 end points of the arcs correspond to a fixed point, $(m, n) \times \mathbb{Z}_2$ and $(m', n') \times \mathbb{Z}_2$. The interior of the arcs correspond to orbits whose stabilizers are (m, n), (m', n') and $\mathbb{Z}_2$. A $\mathbb{Z}_2$ group is generated by, for example, -1×1 in T^2. We denote it by $< -1 \times 1 >$.

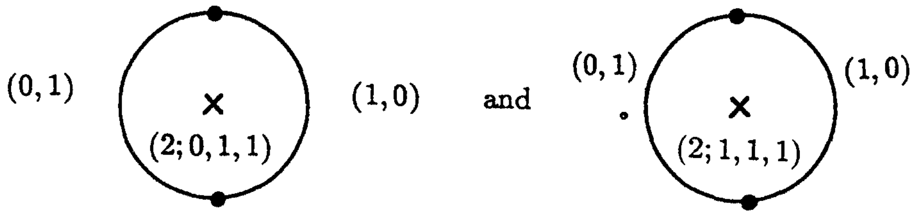
Consider nonorientable 4-manifolds with an effective T^2-action whose orbit spaces are



Then, by results in [K], [OR II] and [Pa], there exists only one manifold corresponding to each orbit space, up to T^2-equvariant diffeomorphism. Note that "weights" changes, in orbit spaces, by reparametrization of T^2 do not affect the total spaces. By the effectiveness and differentiability, the total space is diffeomorphic to one of two manifolds whose orbit spaces are



If we let M_1, M_2 be manifolds corresponding to I and II, according to [B], that the orientable double covers $\tilde{M}_1, \tilde{M}_2$ have the induced T^2-action and their orbit spaces are

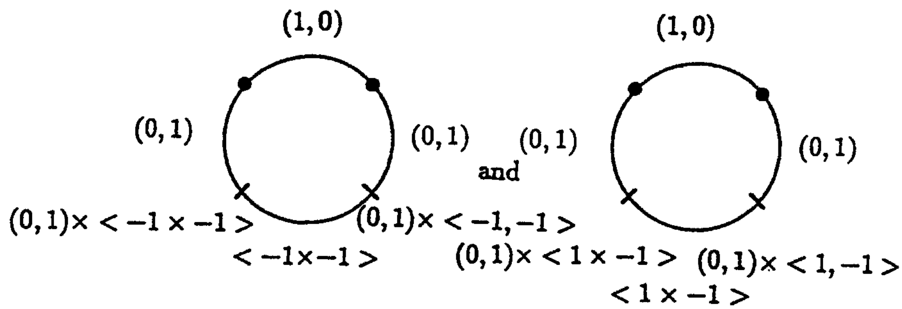


According to Pao, $\tilde{M}_1$ is diffeomorphic to $(S^2 \times S^2)/ < \phi >$ and $\tilde{M}_2$ is diffeomorphic to $(S^2 \times S^2)/ < \psi >$, where ϕ and ψ are involutions of $S^2 \times S^2$ defined by

$$\phi((x,y,z),(x',y',z')) = ((-x,-y,-z),(-x',y',z'))$$
$$\psi((x,y,z),(x',y',z')) = ((-x,-y,-z),(-x',-y',-z'))$$

and $< \phi >$ and $< \psi >$ are $\mathbb{Z}_2$ groups generated by ϕ and ψ respectively.

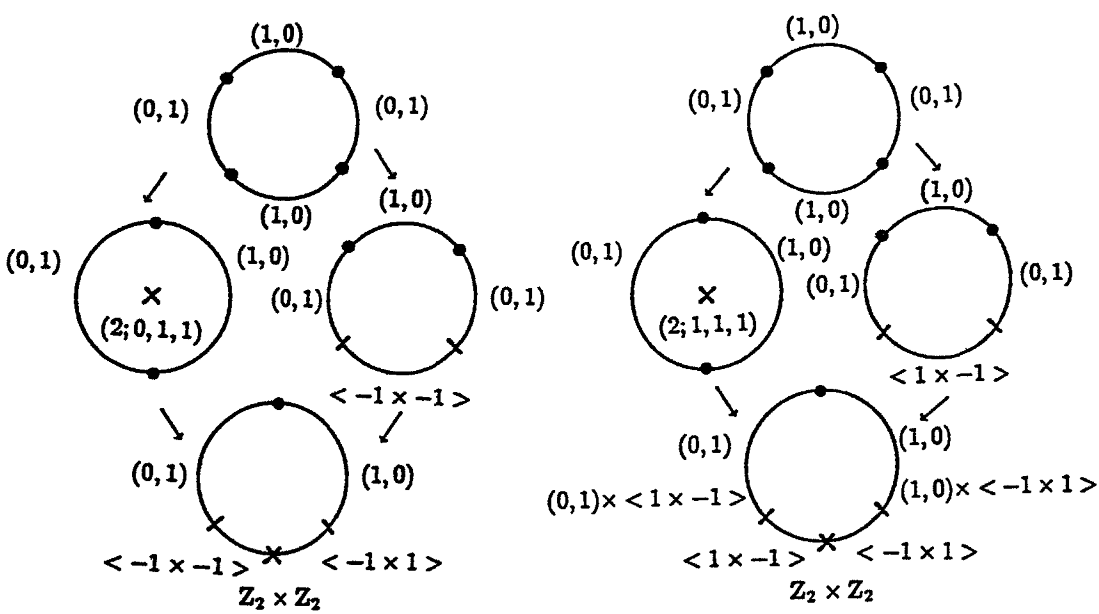
He showed that $\tilde{M}_1$ is not homotopy equivalent to $\tilde{M}_2$, so we can conclude that M_1 is not homotopy equivalent to M_2. It is an interesting fact that there are nonorientable double covers M_1' and M_2' for M_1 and M_2 whose orbit spaces are



From [K] , $M_1' = (S^2 \times S^2)/ < \varphi >$ and $M_2' = (S^2 \times S^2)/ < h >$, where φ and h are involutions of $S^2 \times S^2$ defined by

$$\varphi((x,y,z),(x',y'z')) = ((-x,-y,-z),(-x',-y',z'))$$
$$h((x,y,z),(x',y',z')) = ((-x,-y,-z),(x',y',z')).$$

Furthermore, we have the following diagrams of orbit spaces.

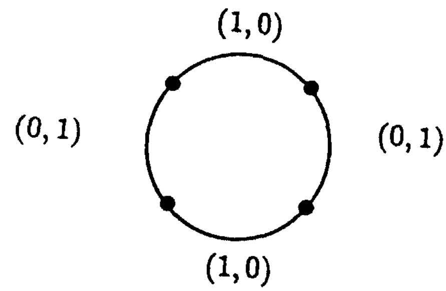


Thus $M_1 = (S^2 \times S^2)/ < \varphi, \phi > M_2 = (S^2 \times S^2)/ < \psi, h >$. We obtain $\pi_1(M_1) = \pi_1(M_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\pi_i(M_1) = \pi_i(M_2) = \mathbb{Z} \times \mathbb{Z}$ for $i \geq 2$ by the homotopy exact sequence. We can give appropriate T^2-actions on M_1 and M_2 which produce the given orbit spaces as follows:

$$T^2 \times S^2 \times S^2 \longrightarrow S^2 \times S^2$$

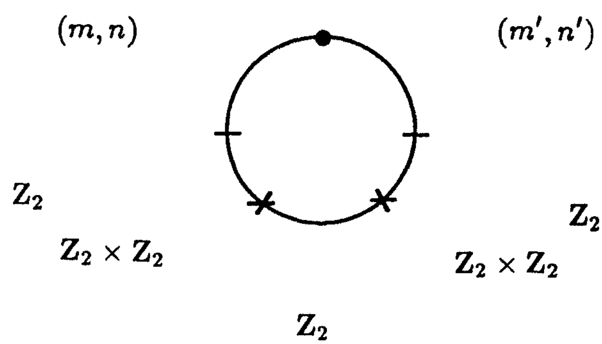
$(\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta))\ \{(\rho_1 \exp(2\pi\imath\theta_1), z_1),\ (\rho_2 \exp(2\pi\imath\theta_2), z_2)\} \longrightarrow \{(\rho_1 \exp(2\pi\imath(\theta_1+\alpha)), z_1),\ (\rho_2 \exp(2\pi\imath(\theta_2+\beta)), z_2)\}$. Then

$(S^2 \times S^2)/T^2 =$

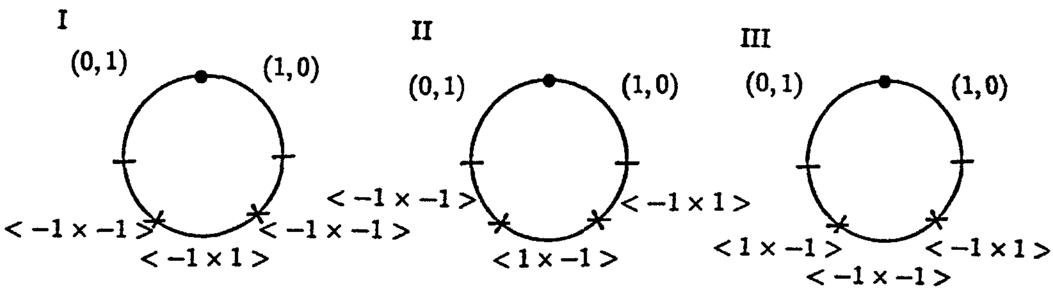


It can be checked that φ, ϕ, h and ψ commute with the T^2-action, and the induced effective T^2-actions on $(S^2 \times S^2)/ < \varphi, \phi >$ *and* $(S^2 \times S^2)/ < h, \psi >$ give the orbit spaces of I and II respectively.

Similarly, by considering closed 4-manifolds with an effective T^2-action whose orbit spaces are of the following type



as in the previous case, the manifolds are diffeomorphic to one of the three manifolds whose orbit spaces are



Let M_1, M_2 and M_3 be the manifolds corresponding to I, II and III respectively. By looking at their orientable covers, and using the results in [Pa](Theorem VI.1), M_1 and M_3 are not homotopy equivalent to M_2, but it is not known whether M_1 is diffeomorphic to M_3 or not. The slice theorem and simple observations prove that $\pi_1(M_1) = \pi_1(M_2) = \pi_1(M_3) = < \alpha, \beta, r \mid \alpha^2 = \beta^2 = r^2 = e, \alpha\beta = \beta\alpha, \beta r = r\beta >$

2. $(\mathrm{N}+1)$-Manifolds with T^{n+1} action

In this section, we are going to show that if, M is a $(n+1)$-dimensional non-orientable manifold with an effective T^n−action, then M is diffeomorphic to $RP^2 \times T^{n-1}$, $K \times T^{n-1}$, $S^{1^\sim} \times S^2 \times T^{n-2}$ or $KS \times T^{n-2}$, $n \geq 2$.

NOTATION 2.1. Let RP^2 be the projective plane. So we denote a point of RP^2 by $[\rho \exp 2\pi\imath\theta, z]$, where [ ] means the equivalence relation. In $S^{1^\sim} \times S^2$, $\sim$ means every point $(\exp 2\pi\imath\theta, \nu)$ in $S^1 \times S^2$ is identified with $(-\exp 2\pi\imath\theta, -\nu)$. Let T^n be the n times product of S^1. K denote the Klein bottle, and KS denote the total space of the non-trivial S^1-principal bundle over K.

When $n = 2$, W. Newman obtained the result (cf. [N]). So we will give a proof by using techniques in [OR] and prove the general case.

LEMMA 2.2. *Let M be a 3-dimensional manifold with an effective T^2-action. Then M is diffeomorphic to $K \times S^1, S^{1^\sim} \times S^2, RP^2 \times S^1$ or KS.*

Proof. By the slice theorem, and by using an automorphism of T^2, we can see that M^* must be one of the following four orbit spaces which are the intervals with "weights".

(i) $\Big|^{(0,1)}_{\langle -1\times 1\rangle}$ (ii) $\Big|^{(0,1)}_{\langle -1\times -1\rangle}$ (iii) $\Big|^{\langle -1\times 1\rangle}_{\langle -1\times 1\rangle}$ (iv) $\Big|^{\langle -1\times 1\rangle}_{\langle -1\times -1\rangle}$

where, on the interior of the intervals, the stabilizer is trivial and at the end points the stabilizer is a circle subgroup or, a Z_2. By applying the same method in [K], we can show that there exist a cross section x such that $\pi \circ x = id$. So, by the cross section theorem (see [K]), we have only to construct spaces with T^2-actions whose orbit spaces are as above.

$$
\begin{aligned}
&\text{(i)}\quad T^2 \times RP^2 \times S^1 \longrightarrow RP^2 \times S^1\\
&\qquad (\exp 2\pi\imath\alpha, \exp 2\pi\imath\beta) \times ([\rho \exp 2\pi\theta, z], \exp 2\pi\imath\phi)\\
&\qquad\qquad \longrightarrow ([\rho \exp 2\pi\imath(\theta+\alpha), z], \exp 2\pi\imath(\theta+\beta))\\
&\text{(ii)}\quad T^2 \times S^{1^\sim} \times S^2 \longrightarrow S^{1^\sim} \times S^2\\
&\qquad (\exp 2\pi\imath\alpha, \exp 2\pi\imath\beta) \times [\exp 2\pi\imath\phi, (\rho \exp 2\pi\imath\theta, z)]\\
&\qquad\qquad \longrightarrow [\exp 2\pi\imath(\phi+\beta), (\rho \exp 2(\theta+\alpha), z)]\\
&\text{(iii)}\; T^2 \times S^{1^\sim} \times S^1 \times S^1 \longrightarrow S^{1^\sim} S^1 \times S^1\\
&\qquad (\exp 2\pi\imath\alpha, \exp 2\pi\imath\beta) \times [x, y, w]\\
&\qquad\qquad \longrightarrow [x \exp 2\pi\imath\alpha, y, w \exp 2\pi\imath\beta]
\end{aligned}
$$

Recall that K is $S^{1^\sim} \times S^1$ where every point (x, y) in $S^1 \times S^1$ is identified with $(-x, \bar{y})$, x,y are complex numbers and $\bar{y}$ is the conjugate of y.

$$
\begin{aligned}
&\text{(iv)}\quad T^2 \times (S^1 \times S^1 \times S^1)/\simeq \longrightarrow (S^1 \times S^1 \times S^1)/\simeq\\
&\qquad (\exp 2\pi\imath\alpha, \exp 2\pi\imath\beta) \times [x, y, w]\\
&\qquad\qquad \longrightarrow [x \exp 2\pi\imath\alpha, y \exp 2\pi\imath\beta, w]
\end{aligned}
$$

Here $\simeq$ means every point (x, y, w) in $S^1 \times S^1 \times S^1$ is identified with $(-x, y, \bar{w}), (-x, -y, -\bar{w})$ and $(x, -y, -w)$.

To finish the proof, it remains to show that $(S^1 \times S^1 \times S^1)/\simeq$ is KS. To show this, we are going to give S^1-action on the space above and obtain a orbit space. Then, by this orbit space and results in [OR], we can see that KS is $(S^1 \times S^1 \times S^1)/\simeq$.

To give a S^1-action, we need the following identification:

$$\psi : (S^1 \times S^1 \times S^1)/\simeq \longrightarrow (S^1 \times S^1 \times S^1)/\sim$$
$$[x, y, w] \longrightarrow [x, y^2, yw]$$

Here, $\sim$ mean every point (x, u, w) in $S^1 \times S^1 \times S^1)$ is identified with $(x, y, y\tilde{w})$.

Then we can give a S^1-action naturally by the complex multiplication on the first coordinate of $(S^1 \times S^1 \times S^1)/\sim$. The orbit space is the Mȍbius strip with Z_2 stabilizer on the boundary. By [OR](see Theorem 5), we can conclude that $(S^1 \times S^1 \times S^1)/\sim$ is KS. This completes the proof.

REMARK. By the homotopy exact sequence, $\pi_1(S^{1^{\sim}} \times S^2) = \mathbb{Z}$. We see that the four spaces above are topologically different, since they have different fundamental groups (see [N]).

Before we go to the general case, we need the following:

Any circle subgroup of T^n can be expressed as

$$\{(x^{a1}, x^{a2}, \cdots x^{an}) \mid x \in S^1\}$$

where the a_i's are relatively prime. We denote the circle subgroup by $(a_1, a_2 \cdots, a_n)$. The follwoing lemma supposes to be well-known.

LEMMA 2.3. *Given a vector $A = (a_1, a_2 \cdots, a_n)$ is $\mathbb{Z}^n$ such that the a_i's are relatively prime, then we can choose*

$$\{(b_{1j}, b_{2j} \cdots, b_{nj}) = B_j | j = 1, 2, \cdots, n-1\}$$

so that $A, B_1, B_2, \cdots, B_{n-1}$ consist of a basis of $\mathbb{Z}^n$, where $\mathbb{Z}^n$ is the n times direct sum of $\mathbb{Z}$.

REMARK. Given a following orbit space

$$\Big|^{(a_1, a_2, \cdots a_n)}_{\mathbb{Z}_2}$$

by lemma 2.3 above, we may assume

$$\Big|^{(1,0,\cdots)}_{\mathbb{Z}_2}$$

where $\mathbb{Z}_2$ is generated by $-1 \times 1 \times \cdots \times 1$ or $1 \times -1 \times 1 \times \cdots \times 1$.

Now we are ready to prove the theorem.

THEOREM 2.4. *Suppose that M is a closed non-orientable $(n+1)$-dimensional differential manifold with an effective T^n-action. Then M is diffeomorphic to $RP^2 \times T^{n-1}, K \times T^{n-2}, S^{1^\sim} \times S^2 \times T^{n-2}$ or $KS \times T^{n-2}, n \geq 3$.*

Proof. By the slice theorem, lemma 2.3, and remark above, we have following four orbit spaces:

$$\text{(i)}\ \Big|^{(1,0,\cdots,0)}_{\langle -1\times 1\times\cdots\times 1\rangle} \quad \text{(ii)}\ \Big|^{(1,0,\cdots,0)}_{\langle 1\times -1\times\cdots\times 1\rangle} \quad \text{(iii)}\ \Big|^{\langle -1\times 1\times\cdots\times 1\rangle}_{\langle -1\times 1\times\cdots\times 1\rangle} \quad \text{(iv)}\ \Big|^{\langle -1\times 1\times\cdots\times 1\rangle}_{\langle 1\times -1\times\cdots\times 1\rangle}$$

Then, as in the lemma 2.2, we can give T^n-actions on the four spaces whose orbit spaces are exactly same as above. This completes the proof.

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