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COMPARISON THEOREMS FOR TUBE VOLUMES IN PRODUCT RIEMANNIAN MANIFOLDS

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1. Introduction

Let $P \subset M$ be an embedding of a compact p-dimensional manifold P to an m-dimensional Riemannian manifold M. We denote by $V_P^M(r)$ the m-dimensional volume of a solid tube of radius r about P and by $A_P^M(r)$ the $(m-1)$-dimensional volume of its boundary. Throughout this paper we assume that $r > 0$ is less than or equal to the distance from P to its nearest focal point. Then we have

$$\int_0^r A_P^M(r)\,dr = V_P^M(r). \tag{1}$$

The well-known Weyl's tube formula for $P \subset \mathbf{R}^m$ can be written as (see for example [2])

$$A_P^{\mathbf{R}^m}(r) = \sum_{c=0}^{[p/2]} \frac{\pi^{(m-p)/2} k_{2c}(R^P)}{2^{c-1}\Gamma((m-p)/2+c)} r^{m-p+2c-1}, \tag{2}$$

where $k_{2c}(R^P)$ are Weyl's curvature invariants constructed from the Riemannian curvature tensor R^P of P. Specifically for an even integer e satisfying $0 \le e \le p$, $k_e(R^P)$ is defined by

$$k_e(R^P) = \int_P I_e(R^P)\,dP,$$

where dP is the volume element of P and $I_e(R^P)$ is given by

$$I_e(R^P) = \frac{1}{2^e(e/2)!} \sum \delta\binom{\alpha}{\beta} R^P_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R^P_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e},$$

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where $\delta\binom{\alpha}{\beta}$ is equal to 1 or -1 according as $\alpha_1, \dots, \alpha_e$ are distinct and an even or odd permutation of $\beta_1, \dots, \beta_e$; and otherwise is equal to zero. The summation is taken over all α and β running from 1 to p. The tube formula for $P \subset E^m(K)$, where $E^m(K)$ is m-dimensional non-Euclidean space of constant curvature K, can be written as ([2])

$$A_P^{E^m(K)}(r) = \sum_{c=0}^{[p/2]} \frac{\pi^{(m-p)/2} k_{2c}(R^P - R^{E^m(K)})}{2^{c-1}\Gamma((m-p)/2+c)} \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{m-p+2c+1} (\cos\sqrt{K}r)^{p-2c}. \tag{3}$$

Here $k_{2c}(R^P - R^{E^m(K)})$ are the same expression as $k_{2c}(R^P)$ except that R^P is replaced by $R^P - R^{E^m(K)}$.

Let $P \subset M$ and $Q \subset N$ be two embeddings, and $P \times Q \subset M \times N$ be the corresponding embedding of the product. Then we have the product formula ([3])

$$A_{P\times Q}^{M\times N}(r) = r\int_0^{\pi/2} A_P^M(r\cos\theta)A_Q^N(r\sin\theta)\,d\theta. \tag{4}$$

In this paper we derive comparison theorems for $A_{P\times Q}^{M\times N}(r)$ and $V_{P\times Q}^{M\times N}(r)$. First we need the following definitions.

For a compact Riemannian manifold P we *define* formally $A_P^{\mathbf{R}^m}(r)$ and $A_P^{E^m(K)}(r)$ by (2) and (3) respectively. If $P \subset M$, we *define* formally ${}^{\mathbf{R}^m}A_P^M(r)$ and ${}^{E^m(K)}A_P^M(r)$ by

$${}^{\mathbf{R}^m}A_P^M(r) = \sum_{c=0}^{[p/2]} \frac{\pi^{(m-p)/2} k_{2c}(R^P - R^M)}{2^{c-1}\Gamma((m-p)/2+c)} r^{m-p+2c-1}, \tag{5}$$

$${}^{E^m(K)}A_P^M(r) = \sum_{c=0}^{[p/2]} \frac{\pi^{(m-p)/2} k_{2c}(R^P - R^M)}{2^{c-1}\Gamma((m-p)/2+c)} \left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{m-p+2c-1} (\cos\sqrt{K}r)^{p-2c}. \tag{6}$$

Here $k_{2c}(R^P - R^M)$ are the same expressions as $k_{2c}(R^P)$ except that R^P is replaced by $R^P - R^M$. We also *define* $V_P^{\mathbf{R}^m}(r)$, $V_P^{E^m(K)}(r)$, ${}^{\mathbf{R}^m}V_P^M(r)$, ${}^{E^m(K)}V_P^M(r)$ by integraing $A_P^{\mathbf{R}^m}(r)$, $A_P^{E^m(K)}(r)$, ${}^{\mathbf{R}^m}A_P^M(r)$, ${}^{E^m(K)}A_P^M(r)$ from 0 to r. These definitions are intrinsic to $P \subset M$ and appear in Gray's comparison theorems [1].

Similarly, if $P \subset M$ and $Q \subset N$, we *define* formally $A_{P\times Q}^{\mathbf{R}^m\times\mathbf{R}^n}(r)$, $A_{P\times Q}^{\mathbf{R}^m\times E^n(K)}(r)$, $A_{P\times Q}^{E^m(K_1)\times E^n(K_2)}(r)$, ${}^{\mathbf{R}^m\times\mathbf{R}^n}A_{P\times Q}^{M\times N}(r)$, ${}^{\mathbf{R}^m\times E^n(K)}A_{P\times Q}^{M\times N}(r)$, ${}^{E^m(K_1)\times E^m(K_2)}A_{P\times Q}^{M\times N}(r)$ by

$$A_{P\times Q}^{\mathbf{R}^m\times\mathbf{R}^n}(r) = r\int_0^{\pi/2} A_P^{\mathbf{R}^m}(r\cos\theta)A_Q^{\mathbf{R}^n}(r\sin\theta)\,d\theta, \tag{7}$$

$$A_{P\times Q}^{\mathbf{R}^m\times E^n(K)}(r) = r\int_0^{\pi/2} A_P^{R^m}(r\cos\theta)A_Q^{E^n(K)}(r\sin\theta)\,d\theta, \tag{8}$$

$$A_{P\times Q}^{E^m(K_1)\times E^n(K_2)}(r) = r\int_0^{\pi/2} A_P^{E^m(K_1)}(r\cos\theta)A_Q^{E^n(K_2)}(r\sin\theta)\,d\theta, \tag{9}$$

$${}^{\mathbf{R}^m\times\mathbf{R}^n}A_{P\times Q}^{M\times N}(r) = r\int_0^{\pi/2} {}^{\mathbf{R}^m}A_P^M(r\cos\theta){}^{\mathbf{R}^n}A_Q^N(r\sin\theta)\,d\theta, \tag{10}$$

$${}^{\mathbf{R}^m\times E^n(K)}A_{P\times Q}^{M\times N}(r) = r\int_0^{\pi/2} {}^{\mathbf{R}^m}A_P^M(r\cos\theta){}^{E^n(K)}A_Q^N(r\sin\theta)\,d\theta, \tag{11}$$

$${}^{E^m(K_1)\times E^n(K_2)}A_{P\times Q}^{M\times N}(r) = r\int_0^{\pi/2} {}^{E^m(K_1)}A_P^M(r\cos\theta){}^{E^n(K_2)}A_Q^N(r\sin\theta)\,d\theta. \tag{12}$$

We also *define* $V_{P\times Q}^{\mathbf{R}^m\times\mathbf{R}^n}(r)$ and ${}^{\mathbf{R}^m\times\mathbf{R}^n}V_{P\times Q}^{M\times N}(r)$ by integrating (7) and (10) respectively.

These intrinsic definitions appear in the following product comparison theorems which generalize Gray's comparison theorems .

THEOREM 1. *Let $P \subset M$ and $Q \subset N$ be two embeddings, and $P \times Q \subset M \times N$ be the corresponding embedding of the product. Assume $r > 0$ is not larger than the distance between $P \times Q$ and its nearest focal point in $M \times N$. Let K^M and K^N be the sectional curvature of M and N respectively.*

(i) *If $K^M > 0$ and $K^N > 0$, then we have*

$$A^{M\times N}_{P\times Q}(r) < {}^{\mathbf{R}^m\times\mathbf{R}^n}A^{M\times N}_{P\times Q}(r) \quad \text{and} \quad V^{M\times N}_{P\times Q}(r) < {}^{\mathbf{R}^m\times\mathbf{R}^n}V^{M\times N}_{P\times Q}(r).$$

(ii) *If $K^M < 0$ and $K^N < 0$, then we have*

$$A^{M\times N}_{P\times Q}(r) > {}^{\mathbf{R}^m\times\mathbf{R}^n}A^{M\times N}_{P\times Q}(r) \quad \text{and} \quad V^{M\times N}_{P\times Q}(r) > {}^{\mathbf{R}^m\times\mathbf{R}^n}V^{M\times N}_{P\times Q}(r).$$

THEOREM 2. *Assume the hypotheses of Theorem 1.*

(i) *If $K^M \geq 0$ and $K^N \geq K$, then*

$$A^{M\times N}_{P\times Q}(r) \leq {}^{\mathbf{R}^m\times E^n(K)}A^{M\times N}_{P\times Q}(r).$$

(ii) *If $K^M \geq K_1$, and $K^N \geq K_2$, then*

$$A^{M\times N}_{P\times Q}(r) \leq {}^{E^m(K_1)\times E^n(K_2)}A^{M\times N}_{P\times Q}(r).$$

(iii) *If $K^M \geq 0$ and $K^N \geq K$, then*

$$A^{M\times N}_{P\times Q}(r) \geq {}^{\mathbf{R}^m\times E^n(K)}A^{M\times N}_{P\times Q}(r).$$

(iv) *If $K^M \leq K_1$ and $K^N \leq K_2$, then*

$$A^{M\times N}_{P\times Q}(r) \geq {}^{E^m(K_1)\times E^n(K_2)}A^{M\times N}_{P\times Q}(r).$$

THEOREM 3. *Under the hypotheses of Theorem* 1 *assume* $\dim P = p \leq 3$ and $\dim Q = q \leq 3$.

(i) *If* $K^M > 0$ and $K^N > 0$ *then*

$$A^{M\times N}_{P\times Q}(r) < {}^{\mathbf{R}^m\times\mathbf{R}^n}A^{M\times N}_{P\times Q}(r) < A^{\mathbf{R}^m\times\mathbf{R}^n}_{P\times Q}(r);$$
$$V^{M\times N}_{P\times Q}(r) < {}^{\mathbf{R}^m\times\mathbf{R}^n}V^{M\times N}_{P\times Q}(r) < V^{\mathbf{R}^m\times\mathbf{R}^n}_{P\times Q}(r).$$

(ii) *If* $K^M \geq 0$ and $K^N \geq K$ *then*

$$A^{M\times N}_{P\times Q}(r) \leq {}^{\mathbf{R}^m\times E^n(K)}A^{M\times N}_{P\times Q}(r) \leq A^{\mathbf{R}^m\times E^n(K)}_{P\times Q}(r).$$

(iii) *If* $K^M \geq K_1$ and $K^N \geq K_2$ *then*

$$A^{M\times N}_{P\times Q}(r) \leq {}^{E^m(K_1)\times E^n(K_2)}A^{M\times N}_{P\times Q}(r) \leq A^{E^m(K_1)\times E^n(K_2)}_{P\times Q}(r).$$

(iv) *If* $K^M < 0$ and $K^N < 0$ *then*

$$A^{M\times N}_{P\times Q}(r) > {}^{\mathbf{R}^m\times\mathbf{R}^n}A^{M\times N}_{P\times Q}(r) > A^{\mathbf{R}^m\times R^n}_{P\times Q}(r);$$
$$V^{M\times N}_{P\times Q}(r) > {}^{\mathbf{R}^m\times\mathbf{R}^n}V^{M\times N}_{P\times Q}(r) > V^{\mathbf{R}^m\times R^n}_{P\times Q}(r).$$

(v) *If* $K^M \leq 0$ and $K^N \leq K$ *then*

$$A^{M\times N}_{P\times Q}(r) \geq {}^{\mathbf{R}^m\times E^n(K)}A^{M\times N}_{P\times Q}(r) \geq A^{\mathbf{R}^m\times E^n(K)}_{P\times Q}(r)$$

(vi) *If* $K^M \leq K_1$ and $K^N \leq K_2$ *then*

$$A^{M\times N}_{P\times Q}(r) \geq {}^{E^m(K_1)\times E^n(K_2)}A^{M\times N}_{P\times Q}(r) \geq A^{E^m(K_1)\times E^n(K_2)}_{P\times Q}(r)$$

Theorem 3 has better formulas for some special cases.

THEOREM 4. *Under the hyposethes of Theorem* 1, *let either* $m = 2$, $p = 0$ *or* $m = 3$, $p = 1$.

(i) *If* $K^M > 0$ and $K^N > 0$ *then* $A^{M\times N}_{P\times Q}(r) < {}^{\mathbf{R}^m}A^M_P(r)^{\mathbf{R}^n}V^N_Q(r)$

(ii) *If* $K^M \geq 0$ and $K^N \geq K$ *then* $A^{M\times N}_{P\times Q}(r) \leq {}^{\mathbf{R}^m}A^M_P(r)^{E^n(K)}V^N_Q(r)$

(iii) *If* $K^M < 0$ and $K^N < 0$ *then* $A^{M\times N}_{P\times Q}(r) > {}^{\mathbf{R}^m}A^M_P(r)^{\mathbf{R}^n}V^N_P(r)$

(iv) *If* $K^M \leq 0$ and $K^N \leq K$ *then* $A^{M\times N}_{P\times Q}(r) \geq {}^{\mathbf{R}^m}A^M_P(r)^{E^n(K)}V^N_Q(r)$.

The product formulas of Lee ([3], p.155 Theorem 4) have the general versions. We also obtain the corresponding comparison theorems.

THEOREM 5. *Assume the hypotheses of Theorem 1. Let* $r_1 \le r_2$ *and* $r = \sqrt{r_1^2 + r_2^2}$. *Write* $A_{P\times Q}^{\mathbf{R}^m \times \mathbf{R}^n} = A_{P\times Q}^{m+n}$ *and* $V_{P\times Q}^{\mathbf{R}^m \times \mathbf{R}^n} = V_{P\times Q}^{m+n}$.

(i) *Let both* p *and* $m+n-q$ *be even. If* $K^M > 0$ *and* $K^N > 0$, *then*

$$
V_{P\times Q}^{m+n}(r) < \begin{cases} \sum_{d=0}^{\infty} V_P^{2d}(r_1) V_Q^{m+n-2d}(r_2) \\ \frac{1}{2\pi r_2} \sum_{d=0}^{\infty} V_P^{2d}(r_1) A_Q^{m+n-2d+2}(r_2) \\ \frac{1}{2\pi r_1} \sum_{d=1}^{\infty} A_P^{2d}(r_1) V_Q^{m+n-2d+2}(r_2) \\ \frac{1}{4\pi^2 r_1 r_2} \sum_{d=1}^{\infty} A_P^{2d}(r_1) A_Q^{m+n-2d+4}(r_2) \end{cases} \tag{13}
$$

and

$$
A_{P\times Q}^{m+n}(r) < \begin{cases} 2\pi r \sum_{d=0}^{\infty} V_P^{2d}(r_1) V_Q^{m+n-2d-2}(r_2) \\ \frac{r}{r_2} \sum_{d=0}^{\infty} V_P^{2d}(r_1) A_Q^{m+n-2d}(r_2) \\ \frac{r}{r_1} \sum_{d=1}^{\infty} A_P^{2d}(r_1) V_Q^{m+n-2d}(r_2) \\ \frac{r}{2\pi r_1 r_2} \sum_{d=1}^{\infty} A_P^{2d}(r_1) A_Q^{m+n-2d+2}(r_2). \end{cases} \tag{14}
$$

(ii) *Let both* p *and* $m+n-q$ *be even. If* $K^M < 0$ *and* $K^N < 0$, *then inequalities (13) and (14) are reversed.*

(iii) *Let both* p *and* $m+n-q$ *be odd. If* $K^M > 0$ *and* $K^N > 0$, *then*

$$
V_{P\times Q}^{m+n}(r) < \begin{cases} \sum_{d=0}^{\infty} V_P^{2d+1}(r_1) V_Q^{m+n-2d-1}(r_2) \\ \frac{1}{2\pi r_2} \sum_{d=0}^{\infty} V_P^{2d+1}(r_1) A_Q^{m+n-2d+1}(r_2) \\ \frac{1}{2\pi r_1} \sum_{d=1}^{\infty} A_P^{2d+1}(r_1) V_Q^{m+n-2d+1}(r_2) \\ \frac{1}{4\pi^2 r_1 r_2} \sum_{d=1}^{\infty} A_P^{2d+1}(r_1) A_Q^{m+n-2d+3}(r_2) \end{cases} \tag{15}
$$

and

$$
A_{P\times Q}^{m+n}(r) < \begin{cases} 2\pi r \sum_{d=0}^{\infty} V_P^{2d+1}(r_1) V_Q^{m+n-2d-3}(r_2) \\ \frac{r}{r_2} \sum_{d=0}^{\infty} V_P^{2d+1}(r_1) A_Q^{m+n-2d-1}(r_2) \\ \frac{r}{r_1} \sum_{d=1}^{\infty} A_P^{2d+1}(r_1) V_Q^{m+n-2d-1}(r_2) \\ \frac{r}{2\pi r_1 r_2} \sum_{d=1}^{\infty} A_P^{2d+1}(r_1) A_Q^{m+n-2d+1}(r_2). \end{cases} \tag{16}
$$

(iv) *Let both p and $m+n-q$ be odd. If $K^M < 0$ and $K^N < 0$, then inequalities* (15) *and* (16) *are reversed.*

(v) *Let p be even and $n-q$ odd. If $K^M > 0$ and $K^N > 0$, then* (13) *and* (14) *hold either for $r_1 < r_2$ or for $r_1 = r_2$ with $m+n-p-q-1 \geq 0$.*

(vi) *Let p be even and $n-q$ odd. If $K^M < 0$ and $K^N < 0$, then the reversed inequalities of* (13) *and* (14) *hold either for $r_1 < r_2$ or for $r_1 = r_2$ with $m+n-p-q-1 \geq 0$.*

(vii) *Let p be odd and $n-q$ even. If $K^M > 0$ and $K^N > 0$, then* (15) *and* (16) *hold either for $r_1 < r_2$ or for $r_1 = r_2$ with $m+n-p-q-1 \geq 0$.*

(viii) *Let p be odd and $n-q$ be even. If $K^M < 0$ and $K^N < 0$, then the reversed inequalities of* (15) *and* (16) *hold either for $r_1 < r_2$ or for $r_1 = r_2$ with $m+n-p-q-1 \geq 0$.*

2. Proofs of Theorems

First we recall Gray's comparison theorems.

THEOREM 6[1]. *Let $P \subset M$ and $r > 0$ be not larger than the distance between P and its nearest focal point.*

(i) *If $K^M \geq 0$ then $A_P^M(r) \leq {}^{\mathbf{R}^m}A_P^M(r)$; $V_P^M(r) \leq {}^{\mathbf{R}^m}V_P^M(r)$.*

(ii) *If $K^M \leq 0$ then $A_P^M(r) \geq {}^{\mathbf{R}^m}A_P^M(r)$; $V_P^M(r) \geq {}^{\mathbf{R}^m}V_P^M(r)$.*

(iii) *If $K^M \geq K$ then $A_P^M(r) \leq {}^{E^m(K)}A_P^M(r)$; $V_P^M(r) \leq {}^{E^m(K)}V_P^M(r)$.*

(iv) *If $K^M \leq K$ then $A_P^M(r) \geq {}^{E^m(K)}A_P^M(r)$; $V_P^M(r) \geq {}^{E^m(K)}V_P^M(r)$.*

When the dimension of a submanifold P is less than or equal to 3 better comparison theorems are given.

COROLLARY 7[1]. *Under the hypotheses of Theorem 6 let $p \leq 3$.*

(i) *If $K^M \geq 0$, then $A_P^M(r) \leq {}^{\mathbf{R}^m}A_P^M(r) \leq A_P^{\mathbf{R}^m}(r)$; $V_P^M(r) \leq {}^{\mathbf{R}^m}V_P^M(r) \leq V_P^{\mathbf{R}^m}(r)$.*

(ii) *If $K^M \leq 0$, then $A_P^M(r) \geq {}^{\mathbf{R}^m}A_P^M(r) \geq A_P^{\mathbf{R}^m}(r)$; $V_P^M(r) \geq {}^{\mathbf{R}^m}V_P^M(r) \geq V_P^{\mathbf{R}^m}(r)$.*

(iii) *If $K^M \geq K$, then $A_P^M(r) \leq {}^{E^m(K)}A_P^M(r) \leq A_P^{E^m(K)}(r)$; $V_P^M(r) \leq {}^{E^m(K)}V_P^M(r) \leq V_P^{E^m(K)}(r)$.*

(iv) *If* $K^M \leq K$, *then* $A_P^M(r) \geq {}^{E^m(K)}A_P^M(r) \geq A_P^{E^m(K)}(r)$; $V_P^M(r) \geq {}^{E^m(K)}V_P^M(r) \geq V_P^{E^m(K)}(r)$.

For the proofs we refer to [1].

Proof of Theorem 1. We prove case (i) only. The same argument with reversed inequalities shows case (ii). It is not difficult to see that if $r > 0$ satisfies the assumption of Theorem 1 then $r > 0$ is also not larger than the distance between P (resp. Q) and its nearest focal point. Since case (i) of Theorem 6 holds true if we replace inequalities by strict inequalities we have from (4)

$$\begin{aligned} A_{P\times Q}^{M\times N}(r) &= r\int_0^{\pi/2} A_P^M(r\cos\theta)A_Q^N(r\sin\theta)\,d\theta \\ &< r\int_0^{\pi/2} {}^{\mathbf{R}^m}A_P^M(r\cos\theta)^{\mathbf{R}^n}A_Q^N(r\sin\theta)\,d\theta \\ &= {}^{\mathbf{R}^m\times\mathbf{R}^n}A_{P\times Q}^{M\times N}(r). \end{aligned}$$

Integrating it with respect to r we obtain $V_{P\times Q}^{M\times N}(r) < {}^{\mathbf{R}^m\times\mathbf{R}^n}V_{P\times Q}^{M\times N}(r)$.

REMARK. In fact

$$\begin{aligned} {}^{\mathbf{R}^m\times\mathbf{R}^n}A_{P\times Q}^{M\times N}(r) = \sum_a\sum_b \frac{\pi^{(m+n-p-q)/2}k_{2a}(R^P - R^M)k_{2b}(R^Q - R^N)}{2^{a+b-1}\Gamma((m+n-p-q)/2+a+b)} \\ r^{m+n-p-q+2a+2b-1}. \end{aligned}$$

Proof of Theorem 2. These are consequences of (4), (8), (9), (10), (11), (12) and Theorem 6.

Proof of Theorem 3 and 4. These are consequences of Theorem 6, Corollary 7, Theorem 1 and Theorem 2.

Proof of Theorem 5. These are consequences of the general version of [3, p.155 Theorem 4], Theorem 1 and Theorem 2.

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