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AFFINELY FLAT 2–TORI WITH IDENTICAL HOLONOMY

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0. Introduction

The study of affine structures on the 2-dimensional torus T^2 was suggested by C. Ehresmann and S. S. Chern, and then has been significantly investigated by many anthors, notably by N. Kuiper [Ku] and T. Nagano and K. Yagi [NY]. But the complete description of the moduli space doesn't seem to be appeared yet.

It is known that the developing maps of affine structures on T^2 are all coverings and there are essentially four different types of developments [NY]. Among these, three convex cases are studied in [Ku], and the remaining non-convex case will be investigated in this paper following the line set up in [Ki]. We are basically looking into the moduli space of linearly flat structures of T^2 with non-trivial covering developments, which is sitting inside the moduli space of affinely flat structures of T^2. We will confine ourselves in this paper to computing and describing the complications arising in the study of the moduli space of the complex linear structures and we defer the study of the moduli space of affine T^2 as a global singular topological object to a subsequent paper.

We study the deformation space or the moduli space on T^2 in terms of holonomy representations, and especially focus on those structures which have the equivalent holonomy representations under the various level of equivalences. This many to one correspondence between the affine structures and the holonomy representations occurs only for the affine structures whose developing maps are non-trivial coverings, and thereby making these affine structures the more interesting part of the moduli space of all affine (or projective) structures on T^2. Since T^2 is the simplest non-trivial affine manifold in a sense, this phenominon

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arising in the study of the moduli space on T^2 should persist through the studies of the moduli space of non-convex affine or projective structures on more general manifolds.

1. The space of affine structures and holonomy representation

An *affinely flat* (or *affine* in short) *structure* on a smooth manifold M is a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$, where $\varphi_\alpha : U_\alpha \to E^n$ is a smooth coordinate chart into the Euclidean n-space, such that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a restriction of an affine transformation on E^n. A *linearly flat* structure is defined similarly replacing affine transformation in the above definition by linear transformation and E^n by $E^n - \{0\}$. Both of these are special cases of (X, A)-structures with $(X, A) = (E^n, \mathrm{Aff}(n, \mathbf{R}))$ and $(X, A) = (E^n - \{0\}, \mathrm{Gl}(n, \mathbf{R}))$. (See [NY], [Kl], [Th], [Ki] for A-structures.)

If we fix a universal covering $p : \tilde{M} \to M$ with the pull back (X, A)-structure, then the space of (X, A)-structures whose developing maps are coverings can be parametrized as follows. Let $D : \tilde{M} \to X$ be a fixed developing map which is also a covering, and let A_D be the group of A-diffeomorphisms (namely, affine or linear diffeomorphisms depending on the context) on $\tilde{M}$ so that we have a short exact sequence of Lie groups,

$$1 \to \Delta \xrightarrow{i} A_D \xrightarrow{\rho} A \to 1, \tag{1.1}$$

where $\rho(a) \in A$ for $a \in A_D$ is the unique affine (or linear) transformation such that $D \circ a = \rho(a) \circ D$, and $\Delta = \ker \rho$ is the deck transformation group of the covering $D : \tilde{M} \to X$. Then any other covering (X, A)-structure will give a developing map $D' : \tilde{M} \to X$ and $D' = D \circ f$ for some lifting $f \in \mathrm{Diffeo}(\tilde{M})$ of id_X such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{f} & \tilde{M} \\ {\scriptstyle D'}\downarrow & & \downarrow{\scriptstyle D} \\ X & \underset{=}{\xrightarrow{\mathrm{id}_X}} & X \end{array}$$

Since f is an A-diffeomorphism and the structure on $\tilde{M}$ is the pull-back structure, $c_f(\tau) = f\tau f^{-1} \in A_D$ for $\tau \in \Pi$, the deck transformation group for $p : \tilde{M} \to M$, i.e., $f \in \mathcal{F}_D = \{f \in \text{Diffeo}(\tilde{M}) \,|\, c_f(\Pi) \subset A_D\}$. Conversely for any $f \in \mathcal{F}_D$, $D \circ f$ defines an A-structure on $\tilde{M}$, where Π acts as A-diffeomorphisms, and thus defines an A-structure on M. If we parametrize (X, A)-structure on M by $f \in \mathcal{F}_D$ denoting it as (M, f), then it is easy to show that id $: (M, f) \to (M, g)$ is an A-diffeomorphism iff there is $a \in A_D$ such that $g = a \circ f$, and that (M, f) and (M, g) are A-diffeomorphic iff there exist $a \in A_D$ and $h \in N(\Pi)$ such that $g \circ h = a \circ f$, where $N(\Pi) =$ the normalizer of Π in Diffeo$(\tilde{M})$. (See [Ki, proposition 2.1].) This suggests us to define a left (resp. right) action of A_D (resp. $N(\Pi)$) as the composition on the left (resp. right), and $A_D \backslash \mathcal{F}_D / N(\Pi)$ is called the *moduli space* of covering A-structure on M. For a given $f \in \mathcal{F}_D$, the composition $\rho \circ c_f : \Pi \to A$ is called the *holonomy representation* of (M, f) and defines a map $\psi : \mathcal{F}_D \to \text{Hom}(\Pi, A)$ with $\psi(f) = \rho \circ c_f$.

If we let $Z(\Pi)$ be the centralizer of Π in Diffeo$(\tilde{M})$, it is easy to check $c : \mathcal{F}_D \to \text{Hom}(\Pi, A_D)$ given by $c(f) = c_f$ induces a injective map : $\mathcal{F}_D / Z(\Pi) \to \text{Hom}(\Pi, A_D)$, also denoted by c. The quotient space $A_D \backslash \mathcal{F}_D / Z(\Pi)$ is called the *deformation space* of covering A-structures on M, and c induces an injective map $A_D \backslash \mathcal{F}_D / Z(\Pi) \to A_D \backslash \text{Hom}(\Pi, A_D)$ and hence an injective map $A_D \backslash \mathcal{F}_D / N(\Pi) \to A_D \backslash \text{Hom}(\Pi, A_D)/\text{Aut}(\Pi)$, where A_D action on $\text{Hom}(\Pi, A_D)$ is by conjugation. (See [Ki] for the details.)

For our case when $M = T^2$, we will fix and use the standard universal covering, $p : \tilde{M} = \mathbf{R}^2 \to M = S^1 \times S^1 \subset \mathbf{C}^2$ given by $p(x, y) = (e^{2\pi i x}, e^{2\pi i y})$. Throughout the paper we will identify $\tilde{M} = \mathbf{R}^2$ with $\mathbf{C}$. We want to study the non-convex affine structures on T^2. As is well known [NY], such affine structure has a non-trivial covering development : $\tilde{M} \to E^2 - \{\text{point}\}$, and may assume that the developing image is $X = E^2 - \{0\}$ by composing with a suitable translation. Hence these are linearly flat structures. Now we will fix a covering development $D : \tilde{M} = \mathbf{C} \to X = \mathbf{C}^* = \mathbf{C} - \{0\}$ given by (the exponential map, $w = e^z = D(z)$. This D certainly defines an affine (or linear) structure on T^2 since Π acts on $\tilde{M} = \mathbf{C}$ as affine transformations. Indeed, if $\tau_1 : z \mapsto z + 1$ and $\tau_2 : z \mapsto z + i$ are the standard generators of Π, then $\rho(\tau_1) : w \mapsto ew$ and $\rho(\tau_2) : w \mapsto e^i w$ are affine (in fact, linear) transformations on E^2. If we let $A = \mathrm{Gl}(2, \mathbf{R})$, then A_D, the group of A-diffeomorphisms on $\tilde{M} = \mathbf{R}^2$ becomes $\widetilde{\mathrm{Gl}}(2, \mathbf{R})$, the universal covering group of $\mathrm{Gl}(2, \mathbf{R})$. Note that $\widetilde{\mathrm{Gl}}(2, \mathbf{R})$ has two components since $\mathrm{Gl}(2, \mathbf{R})$ does. And the deck transformation group Δ for $D : z \mapsto e^z = w$ is an infinite cyclic group generated by $\delta : z \mapsto z + 2\pi i$.

2. Complex linear structures on T^2

Among the linearly flat structures on $T^2 = M$ the complex linear structures constitutes the major portion [NY] and we concentrate to understand the moduli space of these structures. The fixed developing map $D : \tilde{M} = \mathbf{C} \to \mathbf{C}^*$ given by $D(z) = e^z = w$ defines a complex linear structure ($(\mathbf{C}^*, \mathrm{Gl}(1, \mathbf{C}))$-structure or $\mathbf{C}^*$-structure in short) on the torus with holonomy representation ρ given at the end of the previous section. Now all the other complex linear structures can be easily described as follows. Since a translation $g_a : z \mapsto z + a$ on $\tilde{M} = \mathbf{C}$ induces a complex multiplication $m_\alpha : w \mapsto \alpha \cdot w$, $\alpha = e^a$ via D, if we let G_D be the sugbroup of $A_D = \widetilde{Gl}(2, \mathbf{R})$ consisting of translations, then G_D can be identified with $\mathbf{C}$ by identifying g_a with a and $\rho(G_D) \subset A = \mathrm{Gl}(2, \mathbf{R})$ with $\mathbf{C}^*$ by identifying m_α with α. Hence the restriction of (1.1) on G_D becomes

$$
\begin{array}{ccccccccc}
1 & \to & \Delta & \xrightarrow{i} & A_D & \xrightarrow{\rho} & A & \to & 1 \\
 & & \| & & \cup & & \cup & & \\
1 & \to & \Delta & \xrightarrow{i} & G_D & \xrightarrow{\rho} & G & \to & 1 \\
 & & \wr\| & & \| & & \| & & \\
1 & \to & 2\pi i \mathbf{Z} & \xrightarrow{i} & \mathbf{C} & \xrightarrow{\exp} & \mathbf{C}^* & \to & 1
\end{array} \tag{2.1}
$$

This immediately induces the "short exact sequence" of representation varieties,

(2.2)

$$
\begin{array}{ccccccccc}
1 & \to & \mathrm{Hom}(\Pi, \Delta) & \xrightarrow{i_*} & \mathrm{Hom}(\Pi, G_D) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, G) & \to & 1 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
1 & \to & \Delta \times \Delta & \to & \mathbf{C} \times \mathbf{C} & \xrightarrow{\exp \times \exp} & \mathbf{C}^* \times \mathbf{C}^* & \to & 1.
\end{array}
$$

Since a linear transformation $g \in \mathrm{Gl}(2, \mathbf{R})$ on $\tilde{M} = \mathbf{R}^2$ gives rise to a conjugation $c_g : \Pi \to G_D \subset A_D$, $\mathrm{Gl}(2, \mathbf{R}) \subset \mathcal{F}_D$. Furthermore $\mathrm{Gl}(2, \mathbf{R})$

sits in $\mathcal{F}_D/Z(\Pi)$ injectively since $\mathrm{Gl}(2,\mathbf{R}) \cap \mathbf{Z}(\Pi) = 1$. Thus we obtain a diagram of injective maps:

$$\begin{array}{ccc} \mathrm{Gl}(2,\mathbf{R}) & \xrightarrow[1-1]{c} & \mathrm{Hom}(\Pi, G_D) \\ {\scriptstyle 1-1}\downarrow & & \downarrow{\scriptstyle 1-1} \\ \mathcal{F}_D/Z(\Pi) & \xrightarrow[1-1]{c} & \mathrm{Hom}(\Pi, A_D). \end{array} \tag{2.3}$$

In fact, if we identify $\mathrm{Hom}(\Pi, G_D) \cong \mathbf{C} \times \mathbf{C} \cong \mathbf{R}^2 \times \mathbf{R}^2$ with $gl(2,\mathbf{R})$ by writing a pair of vectors in $\mathbf{R}^2$ as two column vectors of a 2×2 matrix, then $c : \mathrm{Gl}(2,\mathbf{R}) \to \mathrm{Hom}(\Pi, G_D) \cong gl(2,\mathbf{R})$ simply becomes an inclusion map. This also shows that

$$c(\mathrm{Gl}(2,\mathbf{R})) = c(\mathcal{F}_D/Z(\Pi)) \cap \mathrm{Hom}(\Pi, G_D). \tag{2.4}$$

Now (2.3) gives us a way of parametrizing the deformation space of complex linear structures as the following induced diagram shows.

$$\begin{array}{ccccc} \mathbf{Z}_2\backslash\mathrm{Gl}(2,\mathbf{R}) & \xrightarrow[1-1]{c} & \mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G_D) & \xrightarrow{\rho_*} & \mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G) \\ {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\ A_D\backslash\mathcal{F}_D/\mathbf{Z}(\Pi) & \xrightarrow[1-1]{c} & A_D\backslash\mathrm{Hom}(\Pi, A_D) & \xrightarrow{\rho_*} & A\backslash\mathrm{Hom}(\Pi, A) \end{array} \tag{2.5}$$

If $a \circ g \in \mathrm{Gl}(2,\mathbf{R})$ for $a \in A_D$ and $g \in \mathrm{Gl}(2,\mathbf{R})$, then $a \in A_D \cap \mathrm{Gl}(2,\mathbf{R})$, and hence the injectivity of the first column map of (2.5) follows from the lemma below.

LEMMA 2.1. *$A_D \cap Gl(2,\mathbf{R}) = \{1, \sigma\}$, where* $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. Observe that $D = \exp : \tilde{M} = \mathbf{C} \to X = \mathbf{C}^*$ sends a line through the origin to an infinite curved spiral except two cases: D (x-axis) = positive x-axis in $\mathbf{C}^*$ and D (y-axis) = unit circle in $\mathbf{C}^*$. Since an element in A_D induces a linear map in X via D, both x-axis and y-axis should be preserved by an element of $A_D \cap \mathrm{Gl}(2,\mathbf{R})$. Moreover by looking at the equation, the only possible such maps are 1 and σ.

The $\mathbf{Z}_2 = \{1, \sigma\}$ action on $\mathrm{Gl}(2, \mathbf{R})$ and on $\mathrm{Hom}(\Pi, G_D)$ (or on $\mathrm{Hom}(\Pi, G)$) are given by complex conjugations, i.e., $\sigma \cdot g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot g$ for $g \in \mathrm{Gl}(2, \mathbf{R})$ and $\sigma \cdot \phi = \bar{\phi}$ for $\phi \in \mathrm{Hom}(\Pi, G_D)$ (or $\mathrm{Hom}(\Pi, G)$), where $\bar{\phi}(\tau) = \overline{\phi(\tau)}$ in the identification $G_D = \mathbf{C}$ (or $G = \mathbf{C}^*$). Note also that $\sigma \cdot \phi = \bar{\phi} = c_\sigma \circ \phi$. Then it is obvious that these two $\mathbf{Z}_2$ actions are equivariant via c, and the injectivity of the map c in the first row of (2.5) follows.

Finally let's show the injectivity of second and third column map of (2.5).

LEMMA 2.2. *Let Z be the center of $A = Gl(2, \mathbf{R})$ and $A^+ = Gl^+(2, \mathbf{R})$.*
(i) $\gamma \in G - Z \Longrightarrow \{\alpha \in A^+ \,|\, \alpha\gamma\alpha^{-1} \in G\} = G$
(ii) $g \in G_D - \rho^{-1}(Z) \Longrightarrow \{a \in A_D^+ \,|\, aga^{-1} \in G_D\} = G_D$.

Proof. If $\gamma \in G - Z$, then γ is of the form $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ with $q \neq 0$. Now it can be easily shown by direct computation that if $\alpha\gamma\alpha^{-1} \in G$ then $\alpha \in G$ or σG. Since $\alpha \in A^+$, $\alpha \in G$. Conversely, if $\alpha \in G$, then $\alpha\gamma\alpha^{-1} = \gamma$ since G is abelian. (ii) follows from (i) by applying $\rho : A_D \to A$.

LEMMA 2.3. (i) *For $\alpha \in A^+$ and $\gamma \in G$, $\alpha\gamma\alpha^{-1} \in G \Rightarrow \alpha\gamma\alpha^{-1} = \gamma$.*
(ii) *For $a \in A_D^+$ and $g \in G_D$, $aga^{-1} \in G_D \Rightarrow aga^{-1} = g$.*

Proof. By Lemma 2.2, if $\gamma \in G - Z$, then $\alpha\gamma\alpha^{-1} \in G$ implies $\alpha \in G$ and hence $\alpha\gamma\alpha^{-1} = \gamma$ since G is abelian. If $\gamma \in Z \subset G$, then clearly $\alpha\gamma\alpha^{-1} = \gamma$. Similarly (ii) follows from Lemma 2.2. Note that $\rho^{-1}(Z)$ is the center of A_D^+ since the translations by real numbers form an identity component $Z_0(A_D)$ of the center of A_D^+ (hence of A_D) and $\rho^{-1}(Z) = \Delta \cdot Z_0(A_D)$.

The above lemma shows that $A^+ \cdot \gamma \cap G = \{\gamma\}$ ($A^+ \cdot r$ is the A^+-orbit of γ with conjugation action, $a \cdot \gamma = a\gamma a^{-1}$) and $A_D^+ \cdot g \cap G_D = \{g\}$.

LEMMA 2.4. (i) *For $\phi \in Hom(\Pi, G)$ and $\alpha \in A^+$, $c_\alpha \circ \phi \in Hom(\Pi, G)$ $\Rightarrow c_\alpha \circ \phi = \phi$.*
(ii) *For $\phi \in Hom(\Pi, G_D)$ and $a \in A_D^+$, $c_a \circ \phi \in Hom(\Pi, G_D) \Rightarrow c_a \circ \phi = \phi$.*

Proof. If $c_\alpha \circ \phi \in \mathrm{Hom}(\Pi, G)$ for $\alpha \in A^+$, then $\alpha\phi(\tau)\alpha^{-1} \in G$ for all $\tau \in \Pi$ and hence $\alpha\phi(\tau)\alpha^{-1} = \phi(\tau)$ by Lemma 2.3. The proof of (ii) follows similarly.

REMARK 2.5. This lemma shows that $A^+ \cdot \phi \cap \mathrm{Hom}(\Pi, G) = \{\phi\}$ for $\phi \in \mathrm{Hom}(\Pi, G)$, where A^+ action is given by $a \cdot \phi = c_a \circ \phi$, and similarly $A_D^+ \cdot \phi \cap \mathrm{Hom}(\Pi, G_D) = \{\phi\}$ for $\phi \in \mathrm{Hom}(\Pi, G_D)$. Since $A^- = \sigma A^+$, $A^- \cdot \phi = \sigma A^+ \cdot \phi$ and hence $A^- \cdot \phi \cap \mathrm{Hom}(\Pi, G) = \{\bar{\phi}\}$ for $\phi \in \mathrm{Hom}(\Pi, G)$. Similarly, $A^- \cdot \phi \cap \mathrm{Hom}(\Pi, G_D) = \{\bar{\phi}\}$ for $\phi \in \mathrm{Hom}(\Pi, G_D)$.

REMARK 2.6. It is clear from Lemma 2.2 that the isotropy group of A^+ action at $\phi \in \mathrm{Hom}(\Pi, G)$ is given by G if $\phi \notin \mathrm{Hom}(\Pi, Z) \subset \mathrm{Hom}(\Pi, G)$ and by A^+ if $\phi \in \mathrm{Hom}(\Pi, Z)$. Similarly, for the isotropy group of A^+ action at $\phi \in \mathrm{Hom}(\Pi, G_D)$, simply replace Z by $Z(A_D^+) = \rho^{-1}(Z) = \Delta \cdot Z_0(A_D)$ and G by G_D in the above.

Now the injectivity of second and third column map of (2.5) follows from Remark 2.5.

The map $\rho_* : \mathrm{Hom}(\Pi, G_D) \to \mathrm{Hom}(\Pi, G)$ is a (product) covering map from (2.2). (See [Ki] for general statement.) But the induced map (also denoted by) $\rho_* : \mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G_D) \to \mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G)$ is an orbifold covering. By (2.4), we see that $\mathbf{Z}_2\backslash\mathrm{Gl}(2, \mathbf{R})$ (or $c(\mathbf{Z}_2\backslash\mathrm{Gl}(2, \mathbf{R}))$ which is an open dense subset of $\mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G_D)$ faithfully parametrize the deformation space of complex linear structures on T^2, and the elements $\rho_* c(\mathbf{Z}_2\backslash\mathrm{Gl}(2, \mathbf{R})) \subset \mathbf{Z}_2\backslash\mathrm{Hom}(\Pi, G)$ are their holonomy representations. Note that $\rho_* c(\mathrm{Gl}(2, \mathbf{R}))$ corresponds exactly to the complement of $S^1 \times S^1 \subset \mathbf{C}^* \times \mathbf{C}^*$ when we identify $\mathrm{Hom}(\Pi, G)$ with $\mathbf{C}^* \times \mathbf{C}^*$, and this in turn implies that $\rho_* c(\mathrm{Gl}(2, \mathbf{R})) = \mathrm{Hom}(\Pi, G) - \mathrm{Hom}(\Pi, S^1)$, where $S^1 \subset \mathbf{C}^* = G$ is the rotation subgroup of G. Therefore for each holonomy representation there corresponds infinitely many affine structures (in fact, all complex linear structures) as elements of the deformation space of linearly flat structures. The following picture shows the developing image of the fundamental domain of three different affine structures of T^2 with identical holonomy representation $\phi = (e^a, e^b) \in \mathbf{C}^* \times \mathbf{C}^* = \mathrm{Hom}(\Pi, G)$.

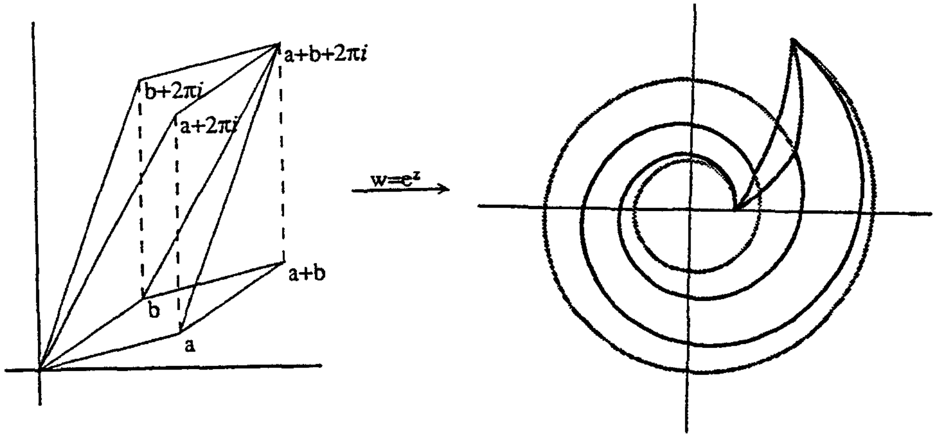

Fig. 1

It shows only three points (a,b), $(a+2\pi i,b)$ and $(a,b+2\pi i)\in \mathbf{C}\times\mathbf{C}$ that correspond to $\phi=(e^a,e^b)$ by ρ_* $(=\exp\times\exp)$, but it already suggests fairly well how all the other possible affine structures parametrized by $(a+2\pi mi, b+2\pi ni)$ look like.

REMARK 2.7. If we work in the category of oriented affine manifolds (A^+-manifolds), then obviously (2.5) becomes a simpler diagram,
(2.6)

$$\begin{array}{ccccc}
Gl(2,\mathbf{R}) & \underset{1-1}{\xrightarrow{c}} & \mathrm{Hom}(\Pi,G_D) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi,G) \\
{\scriptstyle 1-1}\big\downarrow & & {\scriptstyle 1-1}\big\downarrow & & {\scriptstyle 1-1}\big\downarrow \\
A_D^+\backslash\mathcal{F}_D^+/Z(\Pi) & \underset{1-1}{\xrightarrow{c}} & A_D^+\backslash\mathrm{Hom}(\Pi,A_D^+) & \xrightarrow{\rho_*} & A^+\backslash\mathrm{Hom}(\Pi,A^+),
\end{array}$$

where $\mathcal{F}_D^+=\{f\in \mathrm{Diffeo}(\tilde{M})\,|\,c_f:\Pi\to A_D^+\}$.

In this case $\rho_*:\mathrm{Hom}(\Pi,G_D)\to\mathrm{Hom}(\Pi,G)$ is a genuine covering given by (2.2).

Now we summarize the discussions in this section as follows.

THEOREM 2.8. (a) *The deformation space of* $\mathbf{C}^*$*-structures (i.e., complex linear structures - here we identify* G *with* $\mathbf{C}^*$*) on* T^2 *is embedded in the deformation space of* $Gl^+(2,\mathbf{R})$*-structures on* T^2 *and parametrized* *by the space* $\mathcal{T} = \{(a_1, a_2) \in \mathbf{C}^* \times \mathbf{C}^* | a_1/a_2 \notin \mathbf{R}\} \subset \mathbf{C} \times \mathbf{C} \cong$ *Hom*(Π, G_0). ($\mathcal{T} \cong Gl(2, \mathbf{R})$.)

(b) $\mathbf{Z}_2 \backslash \mathcal{T}$ *parametrizes the* $\mathbf{C}^*$*-structures in the deformation space of* $Gl(2, \mathbf{R})$*-structures (i.e., affinely equivalent* $\mathbf{C}^*$*-structures up to diffeomorphisms homotopic to the identity), where* $\mathbf{Z}_2$ *action on* $\mathcal{T}$ *is the complex conjugation.*

(c) *A holonomy representation of a* $\mathbf{C}^*$*-structure lies in Hom*$(\Pi, \mathbf{C}^*)-$*Hom*(Π, S^1), $S^1 \subset \mathbf{C}^*$. *If* $\phi = (\alpha_1, \alpha_2) = (e^{a_1}, a^{a_2}) \in \mathbf{C}^* \times \mathbf{C}^* \cong$ *Hom*$(\Pi, \mathbf{C}^*)$ *is not in Hom*(Π, S^1)*, then* ϕ *is a holonomy representation of the* C^**-structures given by* $\{(a_1, a_2) + 2\pi i(m, n) \,|\, (m, n) \in \mathbf{Z} \times \mathbf{Z}\} \cap \mathcal{T}$.

3. Moduli space of complex linear structures

We start with the diagram,

(3.1)
$$\begin{array}{ccccc}
\mathrm{Gl}(2, \mathbf{R}) & \xrightarrow[1-1]{c} & \mathrm{Hom}(\Pi, G_D) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, G) \\
{\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\
\mathcal{F}_D/Z(\Pi) & \xrightarrow[1-1]{c} & \mathrm{Hom}(\Pi, A_D) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, A)
\end{array}$$

From this, the following diagram is naturally induced.

(3.2)
$$\begin{array}{ccccc}
\mathrm{Gl}(2, \mathbf{R})/\mathrm{Gl}(2, \mathbf{Z}) & \xrightarrow{c} & \mathrm{Hom}(\Pi, G_D)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, G)/\mathrm{Aut}(\Pi) \\
{\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\
\mathcal{F}_D/N(\Pi) & \xrightarrow{c} & \mathrm{Hom}(\Pi, A_D)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)
\end{array}$$

Here all the actions are composition on the right. Let's again identify $\mathrm{Hom}(\Pi, G_D)$ with $\mathbf{C} \times \mathbf{C}$ and $\mathrm{Hom}(\Pi, G)$ with $\mathbf{C}^* \times \mathbf{C}^*$ to carry out the actual computation of ρ_*. Under these identifications $\phi \in \mathrm{Hom}(\Pi, G_D)$ corresponds to $(a_1, a_2) \in \mathbf{C} \times \mathbf{C}$ so that $\phi(\tau_i) : z \mapsto z + a_i$, and $\mathrm{Aut}(\Pi)$ acts as the usual matrix multiplication on the right, i.e., if $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{Aut}(\Pi) = \mathrm{Gl}(2, \mathbf{Z})$, then $\phi \circ \gamma$ corresponds to $(a_1, a_2)\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Similarly, $\phi \in \mathrm{Hom}(\Pi, G)$ corresponds to $(\alpha_1, \alpha_2) \in \mathbf{C}^* \times \mathbf{C}^*$ so that $\phi(\tau_i) : z \mapsto \alpha_i z$, and $\mathrm{Aut}(\Pi)$ acts as the matrix multiplication if we use additive notation for $\mathbf{C}^*$. Since $\rho_* : \mathrm{Hom}(\Pi, G_D) \to \mathrm{Hom}(\Pi, G)$ is an $\mathrm{Aut}(\Pi)$-equivariant covering, we first have to compute the isotropy group $\mathrm{Aut}(\Pi)_\phi$ at each $\phi \in \mathrm{Hom}(\Pi, G)$ and its action on the fiber $\rho_*^{-1}(\phi)$.

Let $\phi = (\alpha_1, \alpha_2) \in \mathbf{C}^* \times \mathbf{C}^* = \mathrm{Hom}(\Pi, G)$ and $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{Gl}(2, \mathbf{Z}) = \mathrm{Aut}(\Pi)$. Write $g = I + \begin{pmatrix} p' & q \\ r & s' \end{pmatrix}$, $p' = p - 1$ and $s' = s - 1$. Then

$$(\alpha_1, \alpha_2) \cdot g = (\alpha_1, \alpha_2) \Longleftrightarrow (\alpha_1, \alpha_2)\begin{pmatrix} p' & q \\ r & s' \end{pmatrix} = 0$$

$$\Longleftrightarrow \det \begin{pmatrix} p' & q \\ r & s' \end{pmatrix} = 0$$

$\Longleftrightarrow (p', r) = m(k, l)$ and $(q, s') = n(k, l)$ for some $m, n \in \mathbf{Z}$, and relatively prime $(k, l) \in \mathbf{Z} \times \mathbf{Z}$. (Note that $\alpha_1 k + \alpha_2 l = 0$ in additive notation)

$$\Longleftrightarrow \begin{pmatrix} p' & q \\ r & s' \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}(m, n)$$

$$\Longleftrightarrow g = \begin{pmatrix} 1 + km & kn \\ lm & 1 + ln \end{pmatrix} = I + \begin{pmatrix} k \\ l \end{pmatrix}(m, n)$$

Now note that $\pm 1 = \det g = km + ln + 1$, and hence $km + ln = 0$ or -2.

Case 1: $km + ln = 0$. In this case, $(m, n) = t(-l, k)$, $t \in \mathbf{Z}$ since (k, l) are relatively prime, and hence $g = I + t\binom{k}{l}(-l, k)$.

Case 2: $km + ln = -2$. Since (k, l) are relatively prime, there exists a pair of integers (u, v) such that $ku + lv = 1$. Then $(m, n) = -2(u, v) + t(-l, k)$, $t \in \mathbf{Z}$ and hence $g = I - 2\binom{k}{l}(u, v) + t\binom{k}{l}(-l, k)$. Let's denote $g_t = I + t\binom{k}{l}(-l, k)$ and $h = I - 2\binom{k}{l}(u, v)$. Then the isotropy group $\mathrm{Aut}(\Pi)_\phi$ is generated by $\langle h, g_t \rangle$ and is isomorphic to the unique non-trivial extension of $\mathbf{Z}$ by $\mathbf{Z}_2$. Note that $h^2 = I$ and any element of $\mathrm{Aut}(\Pi)_\phi$ can be represented as g_t or hg_t.

Now let's examine the action of $\mathrm{Aut}(\Pi)_\phi$ on the fiber $\rho_*^{-1}(\phi)$. Choose any $(a_1, a_2) \in \rho_*^{-1}(\phi)$ with $a_1 k + a_2 l = 0$ (so that $\alpha_1 = e^{a_1}$ and $\alpha_2 = e^{a_2}$ and then $\alpha_1^k \alpha_2^l = 1$). Then all the other elements can be written as $(a_1, a_2) + 2\pi i(m, n)$, $(m, n) \in \mathbf{Z}^2$, and hence

$$\begin{aligned}((a_1, a_2) + 2\pi i(m, n)) \cdot g &= ((a_1, a_2) + 2\pi i(m, n))(I + \begin{pmatrix} k \\ l \end{pmatrix}(m', n')) \\ &= (a_1, a_2) + 2\pi i(m, n) \cdot g\end{aligned}$$

This means that if we identify $(a_1, a_2) + 2\pi i(m, n) \in \rho_*^{-1}(\phi)$ with $(m, n) \in \mathbf{Z}^2$, then $g \in \mathrm{Aut}(\Pi)_\phi \subset \mathrm{Gl}(2, \mathbf{Z})$ acts on the right on $\mathbf{Z}^2$ as a matrix multiplication. Note that $(a_1, a_2) \in \mathbf{C} \times \mathbf{C}$ with $a_1 k + a_2 l = 0$ can be written as $(a_1, a_2) = a(-l, k)$, $a \in \mathbf{C}$ and the corresponding $(\alpha_1, \alpha_2) = (e^{a_1}, e^{a_2}) = (\alpha^{-l}, \alpha^k)$, $\alpha = e^a \in \mathbf{C}^*$. That is $\mathrm{Aut}(\Pi)_\phi$ is the isotropy subgroup $\mathrm{Gl}(2, \mathbf{Z})_{(-l,k)}$ of $\mathrm{Gl}(2, \mathbf{Z})$ which fixes $(-l, k) \in \mathbf{Z}^2$. Furthermore the lattice generated by $(a_1, a_2) = a(-l, k)$ is an infinite cyclic group generated by a (i.e., isomorphic to $\mathbf{Z}$) and hence such $\phi \in \mathrm{Hom}(\Pi, G_D)$ (or $\mathrm{Hom}(\Pi, G)$) is not injective. If $\phi \in \mathrm{Hom}(\Pi, G_D)$ comes from an affine structure, it must be injective. However $\phi \in \mathrm{Hom}(\Pi, G)$ with $\phi(\Pi) = \langle \alpha^{-l}, \alpha^k \rangle \cong \mathbf{Z}$ can be the holonomy group of complex linear structures determined by $a(-l, k) + 2\pi i(m, n) \in \mathbf{C} \times \mathbf{C} = \mathrm{Hom}(\Pi, G)$ for any (m, n) which is not a multiple of $(-l, k)$. See Figure 2 to see what happens geometrically in this case.

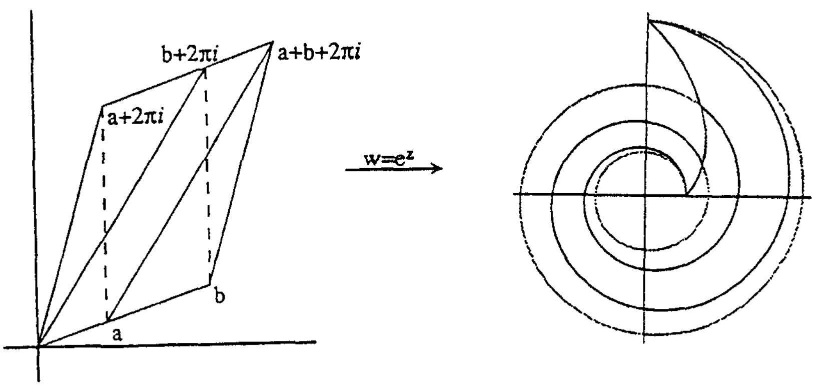

Fig. 2

If we consider the category of oriented affine manifolds, from (2.6) we have the following diagram.

(3.3)

$$\begin{array}{ccccc}
\mathrm{Gl}(2,\mathbf{R})/\mathrm{Gl}(2,\mathbf{Z}) & \xrightarrow[1-1]{c} & \mathrm{Hom}(\Pi, G_D)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, G)/\mathrm{Aut}(\Pi) \\
{\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\
A_D^+\backslash\mathcal{F}_D^+/N(\Pi) & \xrightarrow[1-1]{c} & A_D^+\backslash\mathrm{Hom}(\Pi, A_D^+)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & A^+\backslash\mathrm{Hom}(\Pi, A^+)/\mathrm{Aut}(\Pi)
\end{array}$$

Hence $\mathrm{Hom}(\Pi, G_D)/\mathrm{Aut}(\Pi)$ is sitting inside the moduli space of A^+-structures on T^2 injectively.

Summarizing the discussions so far, we have the following proposition. (In the proposition, we use (p,q) instead of $(-l,k)$.)

PROPOSITION 3.1. (a) *Let* $S = \{\phi \in Hom(\Pi, G) \mid \phi$ *is not injective*$\}$. *Then* S *can be written, with the identification* $Hom(\Pi, G) = \mathbf{C}^* \times \mathbf{C}^*$, *as* $S = \{(\alpha^p, \alpha^q) \in \mathbf{C}^* \times \mathbf{C}^* \mid p, q :$ *relatively prime integers*$\}$.

(b) $Aut(\Pi)$ *acts freely on* $Hom(\Pi, G) - S$. *For* $\phi = (\alpha^p, \alpha^q) \in S$, *with respect to the identification* $Aut(\Pi) = Gl(2, \mathbf{Z})$, *the isotropy group is* $Aut(\Pi)_\phi = \{I - 2\binom{q}{-p}(u,v) + t\binom{q}{-p}(p,q), I + t\binom{q}{-p}(p,q) \mid t \in \mathbf{Z}, \begin{vmatrix} u & p \\ v & q \end{vmatrix} = 1\}$, *and* $Aut(\Pi)_\phi$ *is same as* $Gl(2,\mathbf{Z})_{(p,q)}$, *the isotropy group at* (p,q) *in the usual right action of* $Gl(2,\mathbf{Z})$ *on* $\mathbf{Z}^2$.

(c) $Aut(\Pi)_\phi$ *action on the fiber* $\rho_*^{-1}(\phi) = \{a(p,q) + 2\pi i(m,n) \mid (m,n) \in \mathbf{Z}^2\} \subset \mathbf{C} \times \mathbf{C} = Hom(\Pi, G_D)$ *is given by* $(a(p,q) + 2\pi i(m,n)) \cdot g = a(p,q) + 2\pi i(m,n) \cdot g$, $g \in Aut(\Pi)_\phi = Gl(2,\mathbf{Z})_{(p,q)}$.

REMARK 3.2. This proposition says that for a subgroup $\Gamma \subset G = \mathbf{C}^*$ not in S^1, if $\Gamma \cong \mathbf{Z}^2$, then Γ is the holonomy group of $\mathbf{Z}^2$-many $\mathbf{C}^*$-structures inequivalent as A^+-structures up to diffeomorphisms on T^2, and these structures are given by lattices in $\mathbf{C}$, $L_{(a,b)} = \{ka + lb \mid k, l \in \mathbf{Z}\}$, where $(a,b) = (a_1, a_2) + 2\pi i(m,n)$, for $(m,n) \in \mathbf{Z}^2$ which corresponds to $\Gamma = \langle e^{a_1}, e^{a_2}\rangle$ under the exponential map. But if $\Gamma = \langle e^a \rangle \cong \mathbf{Z}$, then $(a_1, a_2) = a(p,q)$ and some of the lattices $L_{(a,b)}$ are identical by the rule given in (c).

If we consider $\mathbf{C}^*$-structures in the moduli space of $A = \mathrm{Gl}(2, \mathbf{R})$-structures, there is one more complication added by the complex conjugation. The diagram (2.5) or (3.3) induces the diagram of moduli spaces

(3.4)

$$\begin{array}{ccccc}
\mathbf{Z}_2\backslash \mathrm{Gl}(2,\mathbf{R})/\mathrm{Gl}(2,\mathbf{Z}) & \xrightarrow[1-1]{c} & \mathbf{Z}_2\backslash \mathrm{Hom}(\Pi, G_D)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & \mathbf{Z}_2\backslash \mathrm{Hom}(\Pi, G)/\mathrm{Aut}(\Pi) \\
{\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\
A_D\backslash \mathcal{F}_D/N(\Pi) & \xrightarrow[1-1]{c} & A_D\backslash \mathrm{Hom}(\Pi, A_D)/\mathrm{Aut}(\Pi) & \xrightarrow{\rho_*} & A\backslash \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)
\end{array}$$

It is easy to show by a similar calculation we have done before that $\phi \in \mathrm{Hom}(\Pi, G_D)$ and its complex conjugation $\bar{\phi}$ are $\mathrm{Aut}(\Pi)$-equivalent (i.e., gives the same lattice) only when $\phi(\Pi)$ lies in $\mathbf{R}$ or $i\mathbf{R} \subset \mathbf{C} = G_D$. Therefore, on the space of representations arising from the $\mathbf{C}^*$-structures, the two actions do not cross each other. Also note that any two $\mathrm{Gl}(2,\mathbf{R})$-equivalent $\mathbf{C}^*$-structures are necessarily affine-equivalent since the affine equivalence on E^2 in this case has to fix the origin.

Now the following theorem readily follows.

THEOREM 3.3. (a) *The affine equivalence classes of* $\mathbf{C}^*$*-structures on* T^2 *are parametrized by a connected 4-manifold* $\mathbf{Z}_2\backslash Gl(2,\mathbf{R})/Gl(2,\mathbf{Z})$ $(= \mathbf{Z}_2\backslash \mathcal{T}/Gl(2,\mathbf{Z})$ *using the notation of Theorem 2.8).*

(b) *Any free abelian subgroup* Γ *of rank 2 or 1 in* $\mathbf{C}^* = G$ *except rotations (i.e., a subgroup of* $S^1 \subset \mathbf{C}^*$*) can be realized as a holonomy group of a* $\mathbf{C}^*$*-structure on* T^2*, and* Γ *and its complex conjugation* $\bar{\Gamma}$ *are the holonomy groups of the affinely equivalent* $\mathbf{C}^*$*-structures.*

(c) *Those* $\mathbf{C}^*$*-structures (up to affine or* $Gl(2,\mathbf{R})$*-equivalences) giving the same holonomy group are infinite and can be explicitly enumerated from proposition 3.1 (or Remark 3.2) up to complex conjugation.*

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