Comm. Korean Math. Soc. 6(1991), No. 1, pp. 135–143

AN EXTENDED JIANG SUBGROUP OF THE FUNDAMENTAL GROUP OF A TRANSFORMATION GROUP

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F.Rhodes [4] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$ if (G, G) admits a family of preferred paths at e. B.J. Jiang [3] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group of a topological space X.

In this paper, we define an extended Jiang subgroup $J(f, x_0, G)$ which is an extention of Jiang subgroup $J(f, x_0)$ and show some properties of these extended Jiang subgroups.

Let (X, G, π) be a transformation group, where X is a path connected space with x_0 as base point. Given any element g of G, *a path f of order g* with base point x_0 is a continuous map $f : I \longrightarrow X$ such that $f(0) = x_0$ and $f(1) = gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1 + g_1 f_2$ of order $g_1 g_2$ defined by the equations

$$(f_1 + g_1 f_2)(s) = \begin{cases} f_1(2s), & 0 \leq s \leq 1/2 \\ g_1 f_2(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

Two paths f and f' of the same order g are said to be *homotophic* if there is a continuous map $F : I^2 \longrightarrow X$ such that

$$\begin{aligned} F(s,0) &= f(s), && 0 \leq s \leq 1, \\ F(s,1) &= f'(s), && 0 \leq s \leq 1, \\ F(0,t) &= x_0, && 0 \leq t \leq 1, \\ F(1,t) &= gx_0, && 0 \leq t \leq 1. \end{aligned}$$

Received February 5,1991.

This research is supported by The Korea Science and Engineering Foundation research grant.

The homotopy class of a path f of order g was denoted by $[f : g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1 : g_1] * [f_2 : g_2] = [f_1 + g_1f_2 : g_1g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the ***fundamental group*** of (X, G) with base point x_0.

Let f be a self-map of X. A homotopy $H : X \times I \longrightarrow X$ is called a *cyclic homotopy* [3] if $H(x,0) = H(x,1) = f(x)$. This concept of a topological space is generalized to that of a transformation group. A continuous map $H : X \times I \longrightarrow X$ is called an *f-homotopy of order g* if $H(x,0) = f(x)$, $H(x,1) = gf(x)$, where g is an element of G. If H is an f–homotopy of order g, then the path $\alpha : I \longrightarrow X$ given by $\alpha(t) = H(x_0, t)$ will be called the *trace of H*.

DEFINITION. The trace subgroup of f–homotopies of prescribed order $J(f, x_0, G) \subset \sigma(X, f(x_0), G)$ is defined by $J(f, x_0, g) = \{[\alpha : g] \in \sigma(X, f(x_0), G) \mid$ there exists an f–homotopy of order g with trace $\alpha\}$.

In particular, $J(1_X, x_0, G)$ was defined by $E(X, x_0, G)$ in [5] and $J(f, x_0, \{e\})$ was also defined by $J(f, x_0)$ in [3]. From this fact, $J(f, x_0, G)$ will be called by an *extended Jiang subgroup*.

THEOREM 1. *Let X be a pathwise connected CW–complex. If σ is the trace of an f–homotopy of order g and α is homotopic to σ, then α is the trace of an f–homotopy of order g.*

Proof. Let $H : X \times I \longrightarrow X$ be an f–homotopy with trace σ and let h_t be the homotopy connecting σ with α . Let L be the subcomplex of $X \times I$ given by $(X \times 0) \bigcup (X \times 1) \bigcup (x_0 \times I)$. Define a homotopy on L as follows: $k_t : L \longrightarrow X$ such that $k_t(x,0) = f(x)$, $k_t(x,1) = gf(x)$ and $k_t(x_0, s) = h_t(s)$. By the absolute homotopy extension property, there exists a homotopy $K_t : X \times I \longrightarrow X$ such that $K_0 = H$, $K_t|_L = k_t$. Then $K_1 : X \times I \longrightarrow X$ is an f–homotopy of order g with trace α.

THEOREM 2. *$J(f, x_0, G)$ is a subgroup of $\sigma(X, f(x_0), G)$.*

Proof. Let $[\alpha : g_1]$ and $[\beta : g_2]$ be any two element of $J(f, x_0, G)$. Let h_t and k_t be the f–homotopies of order g_1, g_2 with trace α, β respectively.

Define an f–homotopy $\gamma_t : X \longrightarrow X$ such that

$$\gamma_t(x) = \begin{cases} h_{2t}(x), & 0 \leq t \leq 1/2 \\ g_1 k_{2t-1}(x), & 1/2 \leq t \leq 1. \end{cases}$$

Then we have $\gamma_0(x) = f(x)$ and $\gamma_1(x) = g_1 g_2 f(x)$. So the trace of γ_t is the path $\alpha + g_1\beta$ of order $g_1 g_2$. Hence $[\alpha : g_1] * [\beta : g_2] = [\alpha + g_1\beta : g_1 g_2]$ belongs to $J(f, x_0, G)$.

If $[\alpha : g] \in J(f, x_0, G)$, then there exists an f-homotopy h_t of order g such that $h_0 = f$, $h_1 = gf$ and $h_t(x_0) = \alpha(t)$. If we take an f–homotopy of order g^{-1}, $h'_t = g^{-1}h_{1-t} : X \longrightarrow X$, then $h'_0 = f$ and $h'_1 = g^{-1}f$. Since $g^{-1}\alpha\rho$ is the trace of h'_t where $\rho(t)$ is $1 - t$,

$[\alpha : g]^{-1} = [g^{-1}\alpha\rho : g^{-1}]$ belongs to $J(f, x_0, G)$.

REMARK. We know that $E(X, f(x_0), G)$ is a subgroup of $J(f, x_0, G)$. Indeed, let $[\alpha : g]$ be an element of $E(X, f(x_0), G)$. Then there exists a homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = x, H(x, 1) = gx$ and $H(f(x_0), t) = \alpha(t)$. Let $K : X \times I \longrightarrow X$ be an f–homotopy of order g by $K = H \circ (f \times 1_I)$. Then

$$\begin{aligned} K(x,0) &= H \circ (f \times 1_I)(x,0) = H(f(x),0) = f(x) \\ K(x,1) &= H \circ (f \times 1_I)(x,1) = H(f(x),1) = gf(x) \\ K(x_0,t) &= H \circ (f \times 1_I)(x_0,t) = H(f(x_0),t) = \alpha(t). \end{aligned}$$

Thus $[\alpha : g]$ belongs to $J(f, x_0, G)$. This implies

$$E(X, f(x_0), G) \subset J(f, x_0, G).$$

Let X be a pathwise connected CW-complex. In [5], a transformation group (X, G) is called an *H-transformation group* with base point x_0 if there exists a continuous map $\mu : X \times X \longrightarrow X$ such that $\mu(gx_0, x) = \mu(x, gx_0) = gx$ for every element g of G.

COROLLARY 3. *If a transformation group (X, G) is an H-transformation group with base point x_0, then $J(f, x_0, G) = \sigma(X, f(x_0), G)$.*

Proof. In [5], $E(X, f(x_0), G) = \sigma(X, f(x_0), G)$. Since $E(X, f(x_0), G)$ is a subgroup of $J(f, x_0, G)$, $J(f, x_0, G) = \sigma(X, f(x_0), G)$.

Let (X, G) be a transformation group and X^X be the space of all continuous mappings from X to X with compact-open topology. Let G act on X^X continuously by $\pi'(f, g) = gf$. Then (X^X, G, π') is a transformation group .

Let $P : X^X \longrightarrow X$ be the evaluation map given by $P(f) = f(x_0)$. If X is a locally compact,then the evaluation map P is continuous. Since $P(gf) = gf(x_0) = gP(f)$, where $g \in G$ and $f \in X^X, (P, 1_G) : (X^X, G) \longrightarrow (X, G)$ is a category mapping. Thus $P_* : \sigma(X^X, 1_X, G) \longrightarrow \sigma(X, x_0, G)$ is a homomorphism by $P_*[\alpha : g] = [P \circ \alpha : g]$.

REMARK. There is a natural homeomorphism $\phi : (X^X)^I \longrightarrow X^{X \times I}$ given by $\phi(f)(x, s) = f(s)(x)$ for $x \in X$ and $s \in I$.

Note that $f \sim f'$ if and only if $\phi(f) \sim \phi(f')$. Motivated by the following theorem, we can consider $J(f, x_0, G)$ as a generalized evaluation subgroup of the fundamental group of a transformation group (X, G).

THEOREM 4. *Let X be a pathwise connected CW-complex. Then* $P_*\sigma(X^X, f, G) = J(f, x_0, G)$.

Proof. By the above remark, the path $\alpha : I \longrightarrow X^X$ of order g with base point f corresponds to the f–homotopy $\phi(\alpha) : X \times I \longrightarrow X$ of order g. For every element $[\alpha : g] \in \sigma(X^X, f, G)$, $P_*[\alpha : g] = [P \circ \alpha : g]$ and there exists an f–homotopy $\phi(\alpha)$ of order g with trace $P \circ \alpha$. Thus $P_*[\alpha : g] \in J(f, x_0, G)$.

Conversely, for each element $[\alpha : g]$ of $J(f, x_0, G)$, there exists an f–homotopy $F : X \times I \longrightarrow X$ of order g with trace α. Since $\phi : (X^X)^I \longrightarrow X^{X \times I}$ is a homeomorphism such that $\phi(f)(x, s) = (f(s))(x), \phi^{-1}(F)$ is a path of order g with base point f in X^X, for $\phi^{-1}(F) : I \longrightarrow X^X$ such that $\phi^{-1}(F)(0)(x) = F(x, 0) = f(x)$ and $\phi^{-1}(F)(1)(x) = F(x, 1) = gf(x)$. Thus $[\phi^{-1}(F) : g]$ belongs to $\sigma(X^X, f, G)$. Since $P \circ \phi^{-1}(F)(s) = \phi^{-1}(F)(s)(x_0) = F(x_0, s) = \alpha(s)$, we have $[\alpha : g] \in P_*\sigma(X^X, f, G)$. This completes the proof.

The Jiang's result ([3], Lemma 2.1) can be generalized as follows.

THEOREM 5. *Let f and k be self-maps of X.*
(1) $J(k, f(x_0), G) \subset J(k \circ f, x_0, G)$.

(2) If k is a homomorphism of (X, G), i.e., $kg(x) = gk(x)$ for any element g of G, then $k_\pi(J(f, x_0, G)) \subset J(k \circ f, x_0, G)$ where $k_\pi[\alpha : g] = [k\alpha : g]$ for any element $[\alpha : g]$ of $J(f, x_0, G)$.

Proof. (1) Let $[\alpha : g]$ be an element of $J(k, f(x_0), G)$. Then there exists an k–homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = k(x), H(x, 1) = gk(x)$ and $H(f(x_0), t) = \alpha(t)$. Therefore there exists a homotopy $H' = H \circ (f \times 1_I) : X \times I \longrightarrow X$ such that $H'(x, 0) = H(f(x), 0) = kf(x), H'(x, 1) = H(f(x), 1) = gkf(x)$ and $H'(x_0, t) = H(f(x_0), t) = \alpha(t)$. Thus $[\alpha : g]$ belongs to $J(k \circ f, x_0, G)$.
(2) Since $k : (X, G) \longrightarrow (X, G)$ is a homomorphism, k induces a homomorphism $k_\sigma : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, kf(x_0), G)$. Let $[\alpha : g]$ be an element of $J(f, x_0, G)$. Then there exists an f–homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = f(x), H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$. Therefore there exists a homotopy $K = k \circ H : X \times I \longrightarrow X$ such that $K(x, 0) = k \circ H(x, 0) = kf(x)$, $K(x, 1) = k \circ H(x, 1) = kgf(x) = gkf(x)$ and $K(x_0, t) = kH(x_0, t) = k\alpha(t)$.
Thus $k_\sigma[\alpha : g]$ belongs to $J(k \circ f, x_0, G)$. Therefore, we show that

$$k_\sigma(J(f, x_0, G)) \subset J(k \circ f, x_0, G).$$

COROLLARY 6 ([3]). . *Let f and k be self-maps of X. Then*
(1) $J(k, f(x_0)) \subset J(k \circ f, x_0)$,
(2) $k_\pi(J(f, x_0)) \subset J(k \circ f, x_0)$.

If we take a map $i_* : J(f, x_0) \longrightarrow J(f, x_0, G)$ such that $i_*[\alpha] = [\alpha : e]$, then we can identify $J(f, x_0)$ as a subgroup of $J(f, x_0, G)$.

THEOREM 7. *$J(f, x_0)$ is a normal subgroup of $J(f, x_0, G)$.*

Proof. Let $[\alpha : g]$ be any element of $J(f, x_0, G)$ and $[\beta : e]$ be any element of $J(f, x_0)$. Then there exists an f–homotopy $H : X \times I \longrightarrow X$ of order g with trace α and a cyclic homotopy $K : X \times I \longrightarrow X$ such that $K(x_0, t) = \beta$. Define a homotopy

$$F : X \times I \longrightarrow X \quad by$$

$$F(x,t) = \begin{cases} H(x,3t), & 0 \le t \le 1/3, \\ gK(x,3t-1), & 1/3 \le t \le 2/3, \\ H(x,3-3t), & 2/3 \le t \le 1. \end{cases}$$

Then $F(x,0) = H(x,0) = f(x)$, $F(x,1) = H(x,0) = f(x)$ and

$$F(x_0,t) = \begin{cases} \alpha(3t), & 0 \le t \le 1/3, \\ g\beta(3t-1), & 1/3 \le t \le 2/3, \\ \alpha\rho(3t-2), & 2/3 \le t \le 1. \end{cases}$$

Therefore F is a cyclic homotopy such that $F(x_0,t) = (\alpha + g\beta + \alpha\rho)(t)$. So $[\alpha : g] * [\beta : e] * [\alpha : g]^{-1} = [\alpha + g\beta + \alpha\rho : e]$ belongs to $J(f, x_0)$.

In [4], F.Rhodes showed that if λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : \sigma(X, x_0, G) \longrightarrow \sigma(X, x_1, G)$ such that $\lambda_*[\alpha : g] = [\lambda\rho + \alpha + g\lambda : g]$.

THEOREM 8. *Assume that X is a pathwise connected CW-complex. If λ is a path from x_0 to x_1 in X, then the induced homomorphism $(f\lambda)_*$ carries $J(f, x_0, G)$ isomorphically onto $J(f, x_1, G)$.*

Proof. Since $(f\lambda)_* : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, f(x_1), G)$ is an isomorphism, it is sufficient to show that $(f\lambda)_*(J(f, x_0, G)) \subset J(f, x_1, G)$.

Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists an $f-$homotopy $W : X \times I \longrightarrow X$ of order g with trace α. Consider a homotopy $H : (X \times 0) \bigcup (x_1 \times I) \longrightarrow X$ given by $H(x,0) = x$ and$H(x_1,t) = \lambda\rho(t)$. Then there exists a homotopy $\tilde{H} : X \times I \longrightarrow X$ such that $\tilde{H}(x,0) = x$ and $\tilde{H}(x_1,t) = H(x_1,t) = \lambda\rho(t)$. Define $K : X \times I \longrightarrow X$ by

$$K(x,t) = \begin{cases} f\tilde{H}(x,3t), & 0 \le t \le 1/3 \\ W(\tilde{H}(x,1),3t-1), & 1/3 \le t \le 2/3 \\ gf\tilde{H}(x,3(1-t)), & 2/3 \le t \le 1. \end{cases}$$

Then K is an f-homotopy of order g, for

$$K(x_1,t) = \begin{cases} f\tilde{H}(x_1,3t), & 0 \le t \le 1/3 \\ W(\tilde{H}(x_1,1),3t-1), & 1/3 \le t \le 2/3 \\ gf\tilde{H}(x_1,3(1-t)), & 2/3 \le t \le 1 \end{cases}$$
$$= \begin{cases} f\lambda\rho(3t), & 0 \le t \le 1/3 \\ \alpha(3t-1), & 1/3 \le t \le 2/3 \\ gf\lambda(3t-2), & 2/3 \le t \le 1 \end{cases}$$
$$= [f\lambda\rho + \alpha + gf\lambda](t).$$

Thus $(f\lambda)_*([\alpha : g]) = [f\lambda\rho + \alpha + gf\lambda : g]$ belongs to $J(f,x_1,G)$. So, the induced homomorphism $(f\lambda)_*$ is an isomorphism from $J(f,x_0,G)$ to $J(f,x_1,G)$.

THEOREM 9. *If $f,k : X \longrightarrow X$ are homotopic, then $J(f,x_0,G)$ and $J(k,x_0,G)$ are isomorphic.*

Proof. Let $H : X \times I \longrightarrow X$ be a homotopy from f to k and $P(t) = H(x_0,t)$. Then P is a path from $f(x_0)$ to $k(x_0)$. It is sufficient to show that $P_\sigma(J(f,x_0,G)) \subset J(k,x_0,G)$.

Let $[\alpha : g]$ be any element of $J(f,x_0,G)$. Then there exists a homotopy $W : X \times I \longrightarrow X$ such that $W(x,0) = f(x), W(x,1) = gf(x)$ and $W(x_0,t) = \alpha(t)$. If we define a homotopy $K : X \times I \longrightarrow X$ given by

$$K(x,t) = \begin{cases} H(x,1-3t), & 0 \le t \le 1/3 \\ W(x,3t-1), & 1/3 \le t \le 2/3 \\ gH(x,3t-2), & 2/3 \le t \le 1, \end{cases}$$

then $K(x,0) = H(x,1) = k(x)$, $K(x,1) = gH(x,1) = gk(x)$ and

$$K(x_0,t) = \begin{cases} H(x_0,1-3t), & 0 \le t \le 1/3 \\ W(x_0,3t-1), & 1/3 \le t \le 2/3 \\ gH(x_0,3t-2), & 2/3 \le t \le 1. \end{cases}$$

Therefore $[P\rho + \alpha + gP : g]$ belongs to $J(k,x_0,G)$. So $P_\sigma(J(f,x_0,G))$ is contained in $J(k,x_0,G)$.

COROLLARY 10. *If $f, k : X \longrightarrow X$ are homotophic, then $J(f, x_0)$ and $J(k, x_0)$ are isomorphic.*

THEOREM 11. *If $f : (X, G) \longrightarrow (X, G)$ is a homomorphism, i.e., $fg(x) = gf(x)$ for any element g of G and x_1 belongs to $g_0 X_0$ for some $g_0 \in G$, where X_0 is the path connected component of x_0, then $J(f, x_0, G)$ and $J(f, x_1, G)$ are isomorphic.*

Proof. By Theorem 8, we may assume that $x_1 = g_0 x_0$. In [6], the first author proved that $g_0^b : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, g_0 f(x_0), G)$ given by $g_0^b[\alpha : g] = [g_0 \alpha : g_0 g g_0^{-1}]$ is an isomorphism. Therefore, it is sufficient to show $g_0^b(J(f, x_0, G)) \subset J(f, x_1, G)$.

Let $[\alpha : g]$ be an element of $J(f, x_0, G_0)$. Then there exists an f-homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = f(x)$, $H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$. Let $F : X \times I \longrightarrow X$ be a homotopy such that $F = g_0 \circ H \circ (g_0^{-1} \times 1_I)$. Then

$$F(x, 0) = g_0 H(g_0^{-1} x, 0) = g_0 f g_0^{-1}(x) = f(x),$$

$$F(x, 1) = g_0 H(g_0^{-1} x, 1) = g_0 g f g_0^{-1}(x) = g_0 g g_0^{-1} f(x)$$

and

$$F(x_1, t) = g_0 H(g_0^{-1} x_1, t) = g_0 H(x_0, t) = g_0 \alpha(t).$$

Therefore $[g_0 \alpha \; ; \; g_0 g g_0^{-1}]$ belongs to $J(f, x_1, G)$.

THEOREM 12. *If $f : X \longrightarrow X$ is a homeomorphism, k is a self-map of X and $f(x_0) = k(x_0)$, then $J(f, x_0, G)$ is contained in $J(k, x_0, G)$.*

Proof. Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists an f-homotopy $H : X \times I \longrightarrow X$ of order g with trace α. If we define $K : X \times I \longrightarrow X$ be a homotopy such that $K = H \circ (f^{-1} k \times 1_I)$, then

$$K(x, 0) = H(f^{-1} k(x), 0) = k(x),$$
$$K(x, 1) = H(f^{-1} k(x), 1) = gk(x)$$

and

$$\begin{aligned} K(x_0, t) &= H(f^{-1} k(x_0), t) = H(f^{-1} f(x_0), t) \\ &= H(x_0, t) = \alpha(t). \end{aligned}$$

Therefore $[\alpha : g]$ belongs to $J(k, x_0, G)$.

COROLLARY 13. *1) If* $f, k : X \longrightarrow X$ *are homeomorphisms and* $f(x_0) = k(x_0)$, *then* $J(f, x_0, G)$ *is equal to* $J(k, x_0, G)$. *In particular,* $J(f, x_0)$ *is also equal to* $J(k, x_0)$ *for homeomorphisms* f *and* k.

2) If $f : X \longrightarrow X$ *is a homeomorphism and* $f(x_0) = x_0$, *then* $J(f, x_0, G)$ *is equal to* $E(X, x_0, G)$.

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