Comm. Korean Math. Soc. 6(1991), No. 1, pp. 119–133

COMPACT CONTACT CR–SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF A SASAKIAN SPACE FORM*

JAE-BOK JUN, MASUMI KAMEDA AND U-HANG KI

Introduction

The theory of a CR-submanifold of a Sasakian manifold was investigated from two different points of view, namely, one is the case where CR-submanifolds are tangent to the structure vector field, and the other is the case where those are normal to the structure vector field (cf. [11], [12],[13]).

Many subjects for CR-submanifolds in a Sasakian manifold have been studied in [2],[3],[4],[5],[9] and [10] and some interesting results have been obtained. One of which done by Kameda, Ki and Yamaguchi asserts the following :

THEOREM A ([3]). *Let M be a compact totally real submanifold tangent to the structure vector field in a Sasakian space form. If the mean curvature vector is nontrivial and parallel in the normal bundle, and if the induced f-structure in the normal bundle is parallel, then the shape operator in the direction of the mean curvature vector of M is parallel.*

The purpose of the present paper is to investigate compact contact CR-submanifolds in a Sasakian space form, of which the mean curvature vector field is parallel.

In this paper, all manifolds are assumed to be smooth and connected.

Received February 5, 1990.
* Supported by Korea Research Foundation (1986).

1. Submanifolds of a Sasakian manifold

Let $\tilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{\tilde{U} : y^A\}$ and with structure tensor $\{F_B{}^A, G_{CB}, V^A\}$. We then have

$$(1.1)\qquad \begin{cases} F_B{}^D F_D{}^A = -\delta_B^A + V_B V^A, V_B F_A{}^B = 0, F_B{}^A V^B = 0, \\ V^A V_A = 1, G_{BD} F_C{}^B F_A{}^D = G_{CA} - V_C V_A, \end{cases}$$

V_B being the associated 1-form of V^A, where here and in the sequel, the indices $A, B, C, \cdots$ run over the range $\{1, \cdots, 2m+1\}$. Denoting by ∇_B the operator of covariant differentiation with respect to G_{BA}, we also have

$$(1.2)\qquad \nabla_B F_C{}^A = -G_{CB} V^A + \delta_B^A V_C, \nabla_B V^A = F_B{}^A.$$

Let M be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \longrightarrow \tilde{M}$. When the argument is local, M need not be distinguished from $i(M)$. We represent the immersion i locally by $y^A = y^A(x^h)$. Throughout this paper, the indices $h, j, i, \cdots$ run over the range $\{1, \cdots, n+1\}$ and we assume that *the submanifold M of $\tilde{M}$ is tangent to the structure vector field V^A*. If we put $B_j^A = \partial_j y^A$, $\partial_j = \partial/\partial x^j$, then $B_j = (B_j^A)$ are $(n+1)$-linearly independent vectors of M tangent to the submanifold. We choose $2m-n$ mutually orthogonal unit normals $C_x = (C_x^A)$ to M. Since the immersion is isometric, we then have

$$(1.3)\qquad g_{ji} = G_{BA} B_j^B B_i^A,\ g_{xy} = G_{BA} C_x^B C_y^A,\ G_{BA} B_j^B C_x^A = 0,$$

g_{ji} and g_{xy} being the induced metric tensor of M and that of the normal bundle of M respectively, where here and in the sequel the indices x, y, z, u, v, w run over the rang $\{n+2, n+3, \cdots, 2m+1\}$. Therefore, denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation formed with g_{ji}, the equations of Gauss and Weingarten for M are respectively obtained:

$$(1.4)\qquad \nabla_j B_i{}^A = h_{ji}{}^x C_x^A,\ \nabla_j C_x^A = -h_j{}^{ix} B_i^A,$$

where $h_{ji}{}^x$ are the second fundamental forms in the direction of C_x and related by $h_j{}^h{}_x = h_{jix}g^{ih} = h_{ji}{}^y g^{ih} g_{yx}$, $(g^{ji}) = (g_{ji})^{-1}$. The transforms of $B_j{}^A$ and $C_x{}^A$ by F are represented in each coordinate neighborhood as follows:

$$F_B{}^A B_j{}^B = f_j{}^i B_i{}^A + J_j{}^x C_x{}^A, F_B{}^A C_x{}^B = -J_x{}^i B_i{}^A + f_x{}^y C_y{}^A, \tag{1.5}$$

where we have put $f_{ji} = G(JB_j, B_i)$, $J_{jx} = G(JB_j, C_x)$, $J_{xj} = -G(JC_x, B_j)$, $f_{xy} = G(JC_x, C_y)$, $f_j{}^h = f_{ji}g^{ih}$, $J_j{}^x = J_{jy}g^{yx}$ and $f_x{}^y = f_{xz}g^{zy}$, g^{yz} being the contravariant components of g_{yz}. From these definitions we verify that $f_{ji} + f_{ij} = 0$, $J_{jx} = J_{xj}$ and $f_{xy} + f_{yx} = 0$. Since the structure vector V^A is tangent to M, we can also put

$$V^A = v^i B_i^A \tag{1.6}$$

for a vector field v^i on M.

By the properties of the Sasakian structure tensors, it follows, from (1.5) and (1.6) that we have

$$f_j{}^t f_t{}^i = -\delta_j{}^i + v_j v^i + J_j{}^x J_x{}^i, f_x{}^y f_y{}^z = -\delta_x{}^z + J_x{}^t J_t{}^z, \tag{1.7}$$

$$f_j{}^t J_t{}^x + J_j{}^y f_y{}^x = 0, \tag{1.8}$$

$$v^j J_j{}^x = 0, v^j f_j{}^i = 0, v_j v^j = 1. \tag{1.9}$$

Differentiating (1.5) and (1.6) covariantly along M and making use of (1.1), (1.2), (1.4) and these equations, we easily find

$$\nabla_j f_i{}^h = \delta_j{}^h v_i - g_{ji} v^h + h_j{}^h{}_x J_i{}^x - h_{ji}{}^x J_x{}^h, \tag{1.10}$$

$$\nabla_j J_i{}^x = h_{ji}{}^y f_y{}^x - h_{jt}{}^x f_i{}^t, \tag{1.11}$$

$$\nabla_j f_y{}^x = h_{jt}{}^x J_y{}^t - h_{jty} J^{tx}, \tag{1.12}$$

$$\nabla_j v_i = f_{ji}, \tag{1.13}$$

$$h_{jt}{}^x v^t = J_j{}^x. \tag{1.14}$$

In the rest of this section we suppose that the ambient Sasakian manifold $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature c and of real dimension $2m+1$, which is called a Sasakian space form, and is denoted by $\tilde{M}^{2m+1}(c)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned}\tilde{R}_{DCBA} = &\frac{1}{4}(c+3)(G_{DA}G_{CB} - G_{DB}G_{CA})\\ &+\frac{1}{4}(c-1)(V_CV_AG_{DB} - V_CV_BG_{DA} + V_DV_BG_{CA} - V_DV_AG_{CB}\\ &+F_{DA}F_{CB} - F_{DB}F_{CA} - 2F_{DC}F_{BA}).\end{aligned}$$

Thus, we see, using (1.3), (1.5) and (1.6), that equations of the Gauss, Codazzi and Ricci for M are respectively obtained:

$$\begin{aligned}R_{kjih} = &\frac{1}{4}(c+3)(g_{kh}g_{ji} - g_{jh}g_{ki}) + h_{kh}{}^x h_{jix} - h_{jh}{}^x h_{kix}\\ &+\frac{1}{4}(c-1)(v_kv_ig_{jh} - v_jv_ig_{kh} + v_jv_hg_{ki} - v_kv_hg_{ji}\\ &+f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}),\end{aligned} \tag{1.15}$$

$$\nabla_k h_{ji}{}^x - \nabla_j h_{ki}{}^x = \frac{1}{4}(c-1)(J_k{}^x f_{ji} - J_j{}^x f_{ki} - 2J_i{}^x f_{kj}), \tag{1.16}$$

$$\begin{aligned}R_{jiyx} = &\frac{1}{4}(c-1)(J_{jx}J_{iy} - J_{ix}J_{jy} - 2f_{ji}f_{yx})\\ &+ h_{jix}h_i{}^t{}_y - h_{itx}h_j{}^t{}_y,\end{aligned} \tag{1.17}$$

where R_{hjih} and R_{jiyx} are the Riemannian curvature tensor of M and that with respect to the connection induced in the normal bundle of M respectively. We see from (1.15) that the Ricci tensor of M can be expressed as follows:

$$\begin{aligned}R_{ji} = &\frac{1}{4}\{n(c+3) + 2(c-1)\}g_{ji} - \frac{1}{4}(c-1)(n+2)v_jv_i\\ &-\frac{3}{4}(c-1)J_j{}^zJ_{iz} + h^xh_{jix} - h_{jt}{}^xh_i{}^t{}_x\end{aligned} \tag{1.18}$$

with the aid of (1.7), where $h^x = g^{ji}h_{ji}{}^x$.

2. Parallel tensor fields

Let M be a submanifold isometrically immersed in a Sasakian manifold $\tilde{M}$ tangent to the structure vector V. Then M is called a *contact CR-submanifold* ([12])of $\tilde{M}$ if there exists a differentiable distribution $\mathcal{D} : p \longrightarrow \mathcal{D}_p \subset T_p(M)$ on M satisfying the following conditions :

(1) $\mathcal{D}$ is invariant with respect to F, namely, $F\mathcal{D}_p \subset \mathcal{D}_p$ for each point p in M, and
(2) The complementary orthogonal distribution $\mathcal{D}^\perp : p \longrightarrow \mathcal{D}_p^\perp \subset T_p(M)$ is totally real with respect to F, namely, $F\mathcal{D}_p^\perp \subset T_p^\perp(M)$ for each point p in M,

where $T_p(M)$ and $T_p^\perp(M)$ denote the tangent space and normal space respectively at $p \in M$. If $\dim \mathcal{D}_p^\perp = 0$ (resp. $\dim \mathcal{D}_p = 0$), then the contact CR-submanifold M is an invariant submanifold (resp. totally real submanifold) of $\tilde{M}$. If $\dim \mathcal{D}_p^\perp = \dim T_p^\perp(M)$, then M is a generic submanifold of $\tilde{M}$.

By the way, the contact CR- submanifolds of a Sasakian manifold $\tilde{M}$ are characterized as follows:

LEMMA 1.1([12]). *In order for a submanifold M of $\tilde{M}$ to be a contact CR-submanifold, it is necessary and sufficient that*

$$f_j{}^t J_t{}^x = 0 \text{ (equivalently } J_j{}^x f_x{}^y = 0).$$

In such a case, $f_j{}^i$ and $f_y{}^x$ are f-structure in M and that in the normal bundle of M respectively.

A normal vector field $\xi = (\xi^x)$ is called a *parallel section* in the normal bundle if it satisfies $\nabla_j \xi^x = 0$, and furthermore a tensor field S on M is said to be *parallel* in the normal bundle if $\nabla_j S$ vanishes identically.

In this section, the$f$-structure ([7]) in the normal bundle of a contact CR-submanifold is assumed to be parallel. In this case, the equation (1.12) turns out to be

$$h_{jtx} J^{ty} - h_{jt}{}^y J_x{}^t = 0. \tag{2.2}$$

REMARK 1. We notice here that $f_y{}^x$ vanishes identically if M is a generic submanifold of a Sasakian manifold $\tilde{M}$. Thus, a generic submanifold of $\tilde{M}$ has always a trivial f-structure in the normal bundle.

Let H be a mean curvature vector field of M. Namely, it is defined by $H = g^{ji}h_{ji}{}^x C_x/(n+1) = h^x C_x/(n+1)$, which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

From now on we suppose that the mean curvature vector field H of M is nonzero and is parallel in the normal bundle. Then we may choose a local field $\{e_x\}$ in such a way that $H = aC_{n+2}$, where $a = |H|$ in nonzero constant. Because of the choice of the local field, the parallelism of H yields

$$\left[\begin{aligned} &h^x = 0, x \geqq n+3 \\ &h^* = (n+1)a, \end{aligned}\right. \tag{2.3}$$

where here and in the sequel we denote the index $n+2$ by $*$. Since the f-structure in the normal bundle is parallel, it is easily seen form (2.1) that $f_x{}^y \nabla^j J_{jy} = 0$ and hence $h^z f_z{}^y f_{yx} = 0$ by means of (1.11). $f_y{}^x$ being defined the f-structure, it follows that we get $h^z f_z{}^x = 0$, which together with (2.3) gives

$$f_*{}^x = 0. \tag{2.4}$$

because H is nontrivial. Therefore the second equation of (1.7) gives

$$J_{jx}J^{j*} = \delta_x{}^*. \tag{2.5}$$

H being a normal vector field on M, the curvature tensor R_{jiyx} of the connection in the normal bundle shows that R_{ji*x} vanishes identically for any index x. Thus the Ricci equation (1.17) yields

$$h_{jt}{}^x h_i{}^t - h_{it}{}^x h_j{}^t = \frac{1}{4}(c-1)(J_{j*}J_i{}^x - J_{i*}J_j{}^x) \tag{2.6}$$

by means of (2.4), where we have put $h_j{}^k = h_j{}^{k*}$.

For a normal vector field ξ, let A_ξ be a shape operator of the tangent space $T_p(M)$ at p in the direction of ξ, which is defined by $g(A_\xi X, Y) =$

$G(\sigma(X,Y),\xi)$ for any tangent vectors X and Y of $T_p(M)$, where σ denotes the second fundamental form on M.

On the other hand, using (1.7), (2.1) and (2.5), we find

$$|\nabla_k h_{ji} + \frac{1}{4}(c-1)(f_{kj}J_i{}^* + f_{ki}J_j{}^*|^2$$
$$= |\nabla_k h_{ji}|^2 + (c-1)(\nabla_k h_{ji})f^{kj}J^{i*} + \frac{1}{8}(c-1)^2(n - J_{jx}J^{jx}).$$

However, if we take account of (1.7), (1.16) and (2.1), then the second term of the right hand side of above equation is given by $-\frac{1}{4}(c-1)^2(n - J_{jx}J^{jx})$. Thus, it follows that we have

$$(2.7)\quad |\nabla_k h_{ji} + \frac{1}{4}(c-1)(f_{kj}J_i{}^* + f_{ki}J_j{}^*)|^2$$
$$= |\nabla_k h_{ji}|^2 - \frac{1}{8}(c-1)^2(n - J_{jx}J^{jx}).$$

3 Normal f-structure on contact CR-submanifolds

In this section, we assume that the contact CR-submanifold M with parallel f-structure in the normal bundle immersed in a Sasakian space form $\tilde{M}^{2m+1}(c)$ has nontrivial and parallel mean curvature vector.

Furthermore, we suppose that the second fundamental forms σ and the f-structure induced on the submanifold M are commutative to each other, that is, $h_j{}^{tx}f_t{}^h - f_j{}^t h_t{}^{hx} = 0$ for any index x or, equivalently

$$(3.1)\qquad h_{jt}{}^x f_i{}^t + h_{it}{}^x f_j{}^t = 0.$$

In this case, we say that the contact CR-structure induced on M is *normal* ([5]).

Transforming (3.1) by $J_y{}^j f_k{}^i$ and making use of (1.7) and (2.1), we find $h_{jt}{}^x J_y{}^j(\delta_k{}^t - v_k v^t - J_k{}^z J_z{}^t) = 0$, which together with (1.14) gives

$$(3.2)\qquad h_{jt}{}^x J_y{}^t = P_{yz}{}^x J_j{}^z + v_j(\delta_y{}^x + f_y{}^z f_z{}^x),$$

where have put $P_{yz}{}^x = h_{ji}{}^x J_y{}^j J_z{}^i$ and hence it satisfies

$$(3.3)\qquad P_{yz}{}^x f_x{}^w = 0.$$

Denoting $P_{xyz} = g_{zw}P_{xy}{}^{w}$, we see, in a direct consequence of (2.2), that P_{xyz} is symmetric for all indices. When $x = n+2$ in (3.2) we have

$$h_{jt}J_y{}^t = P_{yz*}J_j{}^z + \delta_{y*}v_j \tag{3.4}$$

because of (2.4).

Multiplying $J_z{}^j J_y{}^i$ to (2.6) and summing for j and i, we get

$$P_{yu*}P_z{}^{ux} - P_{zu*}P_y{}^{ux} = \frac{1}{4}(c+3)\{\delta_{z*}J_{jy}J^{jx} - \delta_{y*}J_{jz}J^{jx}\}, \tag{3.5}$$

where we have used (1.9), (2.1), (2.5), (3.2), (3.3) and (3.4). Thus, P_{yzx} being symmetric for all indices, it follows that we obtain

$$P_{zyx}P^{yx*} = P^x P_{zx*} + \frac{1}{4}(c+3)(J_{ix}J^{ix} - 1)\delta_{z*}, \tag{3.6}$$

$$P_{zx}{}^*P_y{}^{z*} = P_{zyx}P^{z**} + \frac{1}{4}(c+3)(J_y{}^iJ_{ix} - \delta_y{}^*\delta_x{}^*), \tag{3.7}$$

where we denoted $P_z{}^{zx} = P^x$.

Defferentiating (3.4) covariantly along M and substituting (1.11) and (1.13), we find

$$\begin{aligned}(\nabla_k h_{jt})J_y{}^t &+ h_j{}^t(h_{kt}{}^z f_{zy} - h_{ksy}f_t{}^s)\\ &= (\nabla_k P_{yz*})J_j{}^z + P_{yz*}(h_{kj}{}^w f_w{}^z - h_{kt}{}^z f_j{}^t) + \delta_{y*}f_{kj},\end{aligned}$$

from which, taking the skew-symmetric part with respect to indices k and j, and using (1.16), (2.6) and (3.1), we obtain

$$\begin{aligned}\frac{1}{4}(c-1)(J_k{}^*f_{jt} - J_j{}^*f_{kt} &- 2J_t{}^*f_{kj})J_y{}^t - 2h_j{}^t h_{ksy}f_t{}^s\\ &= (\nabla_k P_{yz}{}^*)J_j{}^z - (\nabla_j P_{yz}{}^*)J_k{}^z - 2P_{yz}{}^*h_{kt}{}^z f_j{}^t + 2\delta_y{}^* f_{kj},\end{aligned}$$

or, equivalently

$$\begin{aligned}-2h_j{}^t h_{ksy}f_t{}^s &= (\nabla_k P_{yz}{}^*)J_j{}^z - (\nabla_j P_{yz}{}^*)J_k{}^z\\ &\quad - 2P_{yz}{}^*h_{kt}{}^z f_j{}^t + \frac{1}{2}(c+3)\delta_y{}^* f_{kj}.\end{aligned} \tag{3.8}$$

because of (2.1) and (2.5). Transforming (3.8) by $J_x{}^k$ and making use of (2.1), (3.1) and (3.4), we get $\nabla_j P_{xy*} = (J_x{}^t \nabla_t p_{yz*})J_j{}^z$ and hence $\nabla_j P_{yx*} = (J_y{}^t \nabla_t P_{xz*})J_j{}^z$. Thus, the equation (3.8) is reduced to

$$h_j{}^t h_{tsy} f_k{}^s = P_{yz*} h_{jt}{}^z f_k{}^t + \frac{1}{4}(c+3)\delta_{y*} f_{jk},$$

which together with the first equation of (1.7) gives

$$\begin{aligned} & h_{jt} h_s{}^t{}_y(\delta_i{}^s - v_i v^s - J_i{}^z J_z{}^s) \\ &= P_{yz*} h_{jt}{}^z(\delta_i{}^t - v_i v^t - J_i{}^w J_w{}^t) + \frac{1}{4}(c+3)\delta_{y*}(g_{ji} - v_j v_i - J_j{}^z J_{zi}). \end{aligned}$$

By means of (3.2), (3.3) and (3.4), the last equation can be written as

$$\begin{aligned} h_{jt} h_i{}^t{}_y - \delta_{y*} v_j v_i - J_j{}^* J_{iy} = P_{yz}{}^* h_{ji}{}^z & \\ + (P_{zyu} P_v{}^{u*} - P_{vzu} P_y{}^{u*}) J_j{}^v J_i{}^z & \\ + \frac{1}{4}(c+3)\delta_y{}^*(g_{ji} - v_j v_i - J_j{}^z J_{iz}), & \end{aligned} \tag{3.9}$$

which implies

$$h_{ji} h^{ji}{}_y = h^* P_{y**} + \frac{1}{4}(n-1)(c+3)\delta_y{}^* + 2\delta_y{}^*, \tag{3.10}$$

where we have used (1.7), (1.9), (2.3), (2.5), (3.3) and (3.6), which shows that

$$h_2 = h^* P_{***} + \frac{1}{4}(n-1)(c+3) + 2, \tag{3.11}$$

where we have defined $h_2 = h_{ji} h^{ji}$. when $y = n+2$ in (3.9) and make use of (3.7), we find

$$h_{jr} h_i{}^r = P_{z**} h_{ji}{}^z + \frac{1}{4}(c+3)(g_{ji} - v_j v_i - J_{j*} J_{i*}) + v_j v_i + J_{j*} J_{i*},$$

which together with (1.14) and (3.10) yields

$$h_3 = h^* |P_{z**}|^2 + \frac{1}{4}(c+3)(n-2)P_{***} + \frac{1}{4}(c+3)h^* + 3P_{***}, \tag{3.12}$$

where $h_3 = h_{ir}h_i{}^r h^{ji}$.

Making use of (1.7), (2.3) and (3.2), the equation (1.10) implies

$$\nabla_k f_j{}^k = (n - J_{rx}J^{rx})v_j + h^* J_{j*} - P_x J_j{}^x,$$

which implies

$$(3.13)\quad h^{ji}\nabla_k(J_{j*}f_i{}^k) = -h^{kh}h^{ji}f_{jk}f_{ih} + h^* P_{***} - P^x P_{x**} + n - J_{jx}J^{jx},$$

where we have used (1.9), (1.11), (2.4) and (3.4).

By the way, making use of (1.7), (11.4), (3.1) and (3.6), we see that

$$(3.14)\quad h^{hk}h^{ji}f_{kj}f_{hi} = h_2 - P^x P_{x**} - 1 - \frac{1}{4}(c-1)(J_{ix}J^{ix} - 1) - J_{jx}J^{jx}.$$

Therefore (3.13) turns out to be

$$(3.15)\qquad h^{ji}\nabla_k(J_{j*}f_i{}^k) = -\frac{3}{4}(c-1)(n - J_{jx}J^{jx}).$$

Since the submanifold M has parallel mean curvature vector, the Laplacian Δh_{ji} of h_{ji} is given, using the Ricci formula of h_{ji} and (1.16), by

$$(3.16)\qquad \begin{aligned}\Delta h_{ji} &= R_{jr}h_i{}^r - R_{kjih}h^{kh}\\ &\quad + \frac{1}{4}(c-1)\nabla_k(J_*{}^k f_{ji} + J_{j*}f_i{}^k + 2J_{i*}f_j{}^k).\end{aligned}$$

On the other hand, by using (1.14), (2.3) and (2.6), the equation (1.18) implies

$$\begin{aligned}R_{js}h_i{}^s h^{ji} &= \frac{1}{4}\{n(c+3) + 2(c-1)\}h_2 - \frac{1}{4}(c-1)(n+2)\\ &\quad - \frac{3}{4}(c-1)h_s{}^j h^{si}J_{jz}J_i{}^z + h^* h_3 - h_{jr}{}^x h_{isx}h^{rs}h^{ji}\\ &\quad - \frac{1}{4}(c-1)h_j{}^{rx}h^{ji}(J_{r*}J_{ix} - J_{i*}J_{rx}),\end{aligned}$$

which together with (2.5), (3.2), (3.3), (3.4), (3.6) and (3.12) gives

$$
\begin{aligned}
R_{js}h_i{}^s h^{ji} = {} & \frac{1}{4}\{n(c+3)+2(c-1)\}h_2 - \frac{1}{4}(c-1)(n+2) \\
& + \frac{1}{4}(c-1)^2 - \frac{3}{4}(c-1)P^x P_{x**} \\
& - \frac{1}{4}(c-1)(c+2)J_{jx}J^{jx} + |h^* P_{x**}|^2 \\
& + \frac{1}{4}(c+3)(h^*)^2 + 3h^* P_{***} \\
& + \frac{1}{4}(c+3)(n-2)h^* P_{***} - h_{ir}{}^x h_{isx} h^{rs} h^{ji}.
\end{aligned} \tag{3.17}
$$

By means of (1.14), (2.5) abd (3.10), the equation (1.15) gives

$$
\begin{aligned}
R_{kjih}h^{kh}h^{ji} = {} & \frac{1}{4}(c+3)\{(h^*)^2 - h_2\} + \frac{1}{2}(c-1) \\
& + |h^* P_{***}|^2 + \frac{3}{4}(c-1)h^{hk}h^{ji}f_{kj}f_{hi} \\
& - h_{jr}{}^x h_{isx} h^{rs} h^{ji} + \frac{1}{2}(c+3)(n-1)h^* P_{***} \\
& + 4h^* P_{***} + \{\frac{1}{4}(c+3)(n-1)+2\}^2.
\end{aligned} \tag{3.18}
$$

From (3.17) and (3.18) we have

$$
\begin{aligned}
& R_{js}h_i{}^s h^{ji} - R_{kjih}h^{kh}h^{ji} = \{\frac{1}{4}(c+3)n+1\}(h_2 - h^* P_{***}) - \frac{1}{4}(c-1)(n+2) \\
& + \frac{1}{4}(c-1)^2 - \{\frac{1}{4}(c+3)(n-1)+2\}^2 - \frac{1}{2}(c-1) + \frac{3}{16}(c-1)^2(J_{jx}J^{jx}-1) \\
& + \frac{3}{4}(c-1)(J_{jx}J^{jx}+1) - \frac{1}{4}(c-1)(c+2)J_{jx}J^{jx}
\end{aligned}
$$

because of (3.14), which together with (3.11) implies that

$$
R_{js}h_i{}^s h^{ji} - R_{kjih}h^{kh}h^{ji} = \frac{1}{16}(c-1)^2(n - J_{jx}J^{jx}). \tag{3.19}
$$

Multiplying h^{ji} to (3.16) and summing for j and i and taking account of (3.15) and (3.19), we obtain

$$h^{ji} \triangle h_{ji} = -\frac{1}{8}(c-1)^2(n - J_{jx}J^{jx}).$$

Substituting this and (2.7) into the identity :

$$\frac{1}{2} \triangle h_2 = h^{ji} \triangle h_{ji} + | \nabla_k h_{ji}|^2$$

we find

$$\frac{1}{2} \triangle h_2 = | \nabla_k h_{ji} + \frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{i*}f_{kj})|^2.$$

Thus, we have

LEMMA 3.1. *Let M be a compact $(n+1)$-dimensional contact CR-submanifold with nontrivial and parallel mean curvature vector and with parallel f-structure in the normal bundle in a Sasakian space form $\tilde{M}^{2m+1}(c)$. If the f-structure induced on M is normal, then we have*

$$\nabla_k h_{ji} = -\frac{1}{4}(c-1)(J_{j*}f_{ki} + J_{j*}f_{kj}). \tag{3.20}$$

REMARK 2. If M is generic in Lemma 3.1, then we have (3.20).

4. Eigevalues of the shape operator

Let M be a contact CR-submanifold of a Sasakian space form $\tilde{M}^{2m+1}$ (c) satisfying the hypothesis of Lemma 3.1. Furthermore we will consider the case where the second fundamental form in the direction of the mean curvature vector on M is parallel. In the sequel, the shape operator in the direction of C_{n+2} is denoted by A^*. For any constant λ over M, we define a smooth determinant function $det(A^* - \lambda I)$ on M, where I is the identity transformation of the tangent space. Since h_{ji} is parallel, we have $\nabla det(A^* - \lambda I) = 0$. This means that the smooth function $det(A^* - \lambda I)$ is constant over M. From the uniquness of roots, we see that all eigenvalue functions λ_i of A^* are constant. Taking account of the Ricci formular for h_{ji} and the fact that $R_{jix*} = 0$ and $\nabla_k h_{ji} = 0$ we find $(\lambda_j - \lambda_i)R_{jiij} = 0$ for any fixed indices j and i. Thus we have

THEOREM 4.1. *Let M be a compact contact CR-submanifold with nontrivial and parallel mean curvature vector and with parallel f-structure in the normal bundle in a unit sphere $S^{2m+1}(1)$. If the f-structure induced on M is normal and if the eigenvalue functions of the shape operator in the direction of the mean curvature vector are mutually distinct, then M is flat.*

REMARK 3. For a totally real submanifold of a Sasakian space form, Theorem 4.1 is valid [3].

COROLLARY 4.2. *Let M be a compact generic submanifold with nontrivial and parallel mean curvature vector in a unit sphere $S^{2m+1}(1)$. If the f-structure induced on M is normal and if the eigenvalue functions of the shape operator in the direction of the mean curvature vector are mutually distinct, then M is flat.*

Now, let $\mu_1, \cdots, \mu_\alpha$ be mutually distinct eigenvalue of A^* and $n_1, \cdots, n_\alpha$ their multiplicities. Since A^* is parallel, the smooth distribution $T_a (a = 1, \cdots, \alpha)$ which consists of all eigenspaces associated with the eigenvalue can be defined and is parallel. M is assumed to be simply connected and complete, then by means of the de Rham decomposition theorem, the submanifold is a product of Riemannian manifolds $M_1 \times \cdots \times M_\alpha$, where the tangent bundle of M_a corresponds to T_a. Since the shape operator A^* restricted to T_a is propotional to the identity transformation of T_a and each submanifold M_a is totally geodesic in M, the mean curvature vector of M is an umbilical section of M_a in $\tilde{M}^{2m+1}(c)$. Thus, by means of the above arguments and that of Lemma 3.1, we have

THEOREM 4.3.. *Let M be an $(n+1)$-dimensional compact and simply connected contact CR-submanifold with nontrival and parallel mean curvature vector and with parallel f-structure in the normal bundle in a unit sphere $S^{2m+1}(1)$. If the f-structure induced on M is normal, then M is a product of Riemannian manifolds, $M_1 \times \cdots \times M_\alpha$, where α is the number of the distinct eigenvalues of the shape operator in the direction of the mean curvature vector field of M, and the mean curvature vector field of M is an umbilical section of $M_a (a = 1, \cdots, \alpha)$.*

COROLLARY 4.4. *Let M be an $(n+1)$-dimensional compact and simply connected generic submanifold with nontrivial and parallel mean curvature vector in a unit sphere $S^{2m+1}(1)$. If the f-structure induced on M is normal, then we have the same conclusions as those of Theorem 4.3.*

References

1. T. Ikawa and M. Kon, *Remarks on anti-invariant submanifolds of a Sasakian manifold*, Tensor N.S. **30** (1987), 239–246.
2. I. Ishihara, *Anti-invariant submanifolds of a Sasakian space form*, Kodai Math. J. **2** (1979), 171–186.
3. U-H. Ki, M. Kameda and S. Yamaguchi, *Compact totally real submanifolds with parallel mean curvature vector field in a Sasakian space form*, TRU Math. **23** (1987)), 1–15.
4. U-H. Ki and Y. H. Kim, *Generic submanifolds with parallel mean curvature vector of an odd-dimensional sphere*, Kodai Math. J. **4** (1981), 353–370.
5. E. Pak, U-H. Ki, J. S. Pak and Y.H. Kim, *Generic submanifolds with normal f-structure of an odd-dimensional sphere(I)*, J. Korean Math. Soc. **20** (1983), 141–161.
6. S. Yamaguchi, M. Kon and T. Ikawa, *On C-totally real submanifolds*, J. Diff. Geom. **11** (1976), 59–64.
7. K. Yano, *On a structure defined by a tensor field f of type(1,1) satisfying f^3+f* $=0$, Tensor, N.S. **14** (1963), 99–109.
8. K. Yano and M. Kon, *Anti-invariant submanifolds*, Marcel Dekker Inc., 1976.
9. ———, *Anti-invariant submanifolds of a sasakian space form*, Tôhoku Math. J. **29** (1977), 9–23.
10. ———, *Anti-invariant submanifolds of Sasakian space form II*, J. Korean Math. Soc. **13** (1976), 1–14.
11. ———, *Generic submanifolds of Sasakian manifolds*, Kodai Math. J. **3** (1980), 163–196.
12. ———, *Structures on manifolds*, World Scientific Publ., 1984.
13. ———, *On contact CR-submanifolds*, J. Korean Math. Soc. **26** (1989), 231–262.

Department of Mathematics Educations
Kookmin University
Seoul 136-702, Korea
and
Science University of Tokyo
Yamaguchi College
Yamaguchi 756, Japan
and
Department of Mathematics Educations
Kyungpook University
Taegu 702-701, Korea