Comm. Korean Math. Soc. 6(1991), No. 1, pp. 109–118

SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{K}'_e$

DAE HYEON PAHK AND BYUNG KEUN SOHN

S. Sznajder and Z. Zielezny [5,6] showed that the solvability of convolution equations in $\mathcal{K}'_p, p \geq 1$, could characterize the local growth condition of the Fourier transform of the given convolutor. In this paper we study the same problem in the space $\mathcal{K}'_e$. In other words, let $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ be the space of convolution operators in $\mathcal{K}'_e$. Under what conditions on $S \in O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ is $S * \mathcal{K}'_e = \mathcal{K}'_e$? The last equality means that the mapping $u \to S * u$ maps $\mathcal{K}'_e$ onto $\mathcal{K}'_e$. We found the following one sufficient condition and one necessary condition for the solvability of the convolution equation

$$(1) \qquad S * u = v$$

in $\mathcal{K}'_e$:

THEOREM 1. *If S is a distribution in $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ and there exist positive constants c and N such that*

$$(2) \qquad \sup_{\substack{|z| \leq M(\frac{1}{2}\log(\log(10+|\xi|))) \\ z \in C^n}} |\hat{S}(z+\xi)| \geq \frac{c}{(1+|\xi|)^N}$$

for all $\xi \in R^n$, then $S * \mathcal{K}'_e = \mathcal{K}'_e$.

THEOREM 2. *If S is a distribution in $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ satisfying $S * \mathcal{K}'_e = \mathcal{K}'_e$, then there exist constants c and N such that*

$$(3) \qquad \sup_{\substack{|z| \leq M(\frac{1}{2}\log(10+|\xi|)) \\ z \in C^n}} |\hat{S}(z+\xi)| \geq \frac{c}{(1+|\xi|)^N}$$

for all $\xi \in R^n$.

Received January 28, 1991.
Supported in part by KOSEF and Ministry of Education.
Supported in part by Injae RSF.

REMARK. We expect the condition (3) will be the necessary and sufficient condition for the solvability of the convolution equation (1) in $\mathcal{K}'_e$. But we did not succeed yet.

Before presenting the proofs we state the basic facts about the spaces $\mathcal{K}'_e$ and $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$; for the proofs we refer to [3,4].

We denote by $\mathcal{K}_e$ the space of all functions $\varphi \in C^\infty(R^n)$ such that

$$\nu_k(\varphi) = \sup_{x \in R^n, |\alpha| \le k} e^{M(k|x|)} |D^\alpha \varphi(x)| \quad < \infty, \quad k = 0, 1, 2, \cdots$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$D^\alpha = (\frac{1}{i}\frac{\partial}{\partial x_1})^{\alpha_1}(\frac{1}{i}\frac{\partial}{\partial x_2})^{\alpha_2} \cdots (\frac{1}{i}\frac{\partial}{\partial x_n})^{\alpha_n} \text{ and } M(x) = e^x - x - 1.$$

The topology in $\mathcal{K}_e$ is defined by the family of semi-norms ν_k. Then $\mathcal{K}_e$ becomes a Frechet space.

The dual $\mathcal{K}'_e$ of $\mathcal{K}_e$ is a space of distributions. A distribution u is in $\mathcal{K}'_e$ if and only if there exists a multi-index α, an integer $k \ge 0$ and a bounded, continuous function f on R^n such that

$$u = D^\alpha[\exp(M(k|x|))f(x)].$$

If $u \in \mathcal{K}'_e$ and $\varphi \in \mathcal{K}_e$, then the convolution $u * \phi$ is a function in $C^\infty(R^n)$ defined by

$$u * \varphi(x) = < u_y, \varphi(x-y) >,$$

where $< u, \varphi >= u(\varphi)$.

The space $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ of convolution operators in $\mathcal{K}'_e$ consists of distributions $S \in \mathcal{K}'_e$ satisfying one of the following equivalent conditions :

(i) The products $S_x \exp(M(k|x|)), k = 0, 1, 2, \cdots$ are tempered distributions.

(ii) For every $k \ge 0$ there exists an integer $m \ge 0$ such that

$$S = \sum_{|\alpha| \le m} D^\alpha f_\alpha.$$

where f_α, $|\alpha| \le m$, are continuous functions in R^n whose products with $\exp(M(k|x|))$ are bounded.

(iii) For every $\varphi \in \mathcal{K}'_e$ the convolution $S * \varphi$ is in $\mathcal{K}'_e$; moreover, the mapping $\varphi \to S * \varphi$ of $\mathcal{K}_e$ into $\mathcal{K}_e$ is continuous.

If $S \in O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$ and $\check{S}$ is the distribution in $\mathcal{K}'_e$ defined by $< \check{S}, \varphi >=$ $< S_x, \varphi(-x) >, \phi \in \mathcal{K}_e$, then $\check{S}$ is also in $O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$. The convolution of S with $u \in \mathcal{K}'_e$ is then defined by

$$(4) \qquad < S * u, \varphi >=< u * S, \varphi >=< u, \check{S} * \varphi >, \quad \varphi \in \mathcal{K}_e.$$

For a function $\varphi \in \mathcal{K}_e$, the Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^n} e^{-i<\xi,x>} \varphi(x) dx$$

can be continued in C^n as an entire function such that

$$(5) \qquad w_k(\hat{\varphi}) = \sup_{\zeta \in C^n} (1 + |\zeta|)^k e^{-\Omega(\frac{|\zeta|}{k})} |\hat{\varphi}(\zeta)| < \infty, \quad k = 1, 2, \cdots$$

where $\zeta = \xi + i\eta$ and $\Omega(x) = (|x|+1)\log(|x|+1) - |x|$ is the Young's dual function of $M(x)$. We denote by K_e the space of Fourier transforms of functions in $\mathcal{K}_e$. If the topology in K_e is defined by the family of seminorms w_k, then the Fourier transformation is an isomorphism of $\mathcal{K}_e$ onto K_e.

The dual K'_e of K_e is the space of Fourier transforms of distributions in $\mathcal{K}'_e$. The Fourier transform $\hat{u}$ of a distribution $u \in \mathcal{K}'_e$ is defined by the Parseval formula ·

$$< \hat{u}, \hat{\varphi} >= (2\pi)^n < u_x, \varphi(-x) > .$$

For $S \in O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$, the Fourier transform $\hat{S}$ is a function which can be continued in C^n as an entire function having the following property : For every $k > 0$ there exists constants c and N such that

$$(6) \qquad |\hat{S}(\xi + i\eta)| \le c(1 + |\zeta|)^N e^{\Omega(\eta/k)}.$$

Futhermore, if $S \in O'_c(\mathcal{K}'_e, \mathcal{K}'_e)$, and $u \in \mathcal{K}'_e$, we have the formula

$$(7) \qquad \widehat{S * u} = \hat{S}\hat{u},$$

where the product on the right-hand side is defined in K'_e by $< \hat{S}\hat{u}, \psi >=$ $< \hat{u}, \hat{S}\psi >, \psi \in K_e$.

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (See [3], lemma 3.2):

If F, G and F/G are entire functions and r be an arbitrary positive number, then

$$|F(\zeta)/G(\zeta)| \le (\sup_{|\zeta - z|<4r} |F(z)|)(\sup_{|\zeta - z|<4r} |G(z)|)/(\sup_{|\zeta - z|<r} |G(z)|)^2$$

where ζ, $z \in C^n$.

Proof of the theorem 1. Let $T = \check{S}$, then T satisfies the hypotheses of the theorem. Define the map, for given $v \in \mathcal{K}'_e$, $L_v : T * \mathcal{K}_e \to C$ by $L_v(T * \varphi) =$
$< v, \varphi >$ for every $\varphi \in \mathcal{K}_e$. Then L_v is a linear map on the subspace $T * \mathcal{K}_e$ of $\mathcal{K}_e$. In order to prove the theorem it suffices to show that L_v is continuous on $T * \mathcal{K}_e$. Indeed , we can extend L_v, by the Hahn-Banach theorem, to a continous linear functional on $\mathcal{K}_e$, i.e. to a distribution in $\mathcal{K}'_e$, call it u. Then $< S * u, \varphi >=< u, T * \varphi >= L_v(T * \varphi) =< v, \varphi >$ for all $\varphi \in \mathcal{K}_e$. Hence $S * u = v$ and so $S * \mathcal{K}'_e = \mathcal{K}'_e$. Futhermore, since the continuity of L_v is equivalent to that of the map $T*\varphi \to \varphi : T*\mathcal{K}_e \to \mathcal{K}_e$ and the Fourier transform is an isomorphism from $\mathcal{K}_e$ onto K_e, it suffices to prove the equivalent statement that the mapping $\hat{T}\hat{\varphi} \to \hat{\varphi} : \hat{T}K_e \to K_e$ is continuous.

Suppose that $\hat{T}\hat{\varphi} = \hat{\psi}$, where $\hat{\varphi}, \hat{\psi} \in K_e$. We recall that $\hat{T}$ is an entire function satisfying condition (2) and an estimate, for given $\epsilon > 0$, of the form (6):

$$|\hat{T}(\zeta)| \le c_\epsilon (1 + |\zeta|)^{N_1} \exp(\Omega(\epsilon\eta)), \qquad \zeta \in C^n,$$

for some constant c and N_1. Setting

$$r = M(\frac{1}{2}\log(\log(10 + |\zeta|))) + |\eta| \tag{8}$$

and making use of the inequality

$$\Omega(x) + \Omega(y) \le \Omega(x + y) \le \Omega(2x) + \Omega(2y), \qquad \text{for } x, y \in R,$$

we obtain

(9)

$$\begin{aligned}
&\sup_{\substack{|\zeta-z|<4r\\ z\in C^n}} |\hat{T}(z)| = \sup_{|z|<4r} |\hat{T}(z+\zeta)| \\
&\leq \sup_{|z|<4r} c_\epsilon (1+|z+\zeta|)^{N_1} \exp(\Omega(\epsilon(y+\eta))) \\
&\leq c_\epsilon \sup_{|z|<4r} (1+|z|+|\zeta|)^{N_1} \exp(\Omega(2\epsilon y)) \exp(\Omega(2\epsilon\eta)) \\
&\leq c_\epsilon\{1+|\zeta|+4M(\frac{1}{2}\log(\log(10+|\xi|)))+4|\eta|\}^{N_1} \exp(\Omega(2\epsilon\eta)) \\
&\qquad \exp[\Omega(8\epsilon M(\frac{1}{2}\log(\log(10+|\xi|)))+8\epsilon|\eta|)] \\
&\leq c_\epsilon(1+|\xi|+5|\eta|+4(\log(10+|\xi|))^{1/2})^{N_1} \exp(\Omega(2\epsilon\eta)) \\
&\qquad \exp[\Omega(16\epsilon M(\frac{1}{2}\log(\log(10+|\xi|))))] \exp(\Omega(16\epsilon\eta)) \\
&\leq c'_\epsilon(1+|\xi|)^{N_1}(1+|\eta|)^{N_1} \exp(\Omega(18\epsilon\eta)) \exp(16\epsilon\Omega((\log(10+|\epsilon|))^{1/2})) \\
&\leq c'_\epsilon(1+|\xi|)^{N_1}(1+|\eta|)^{N_1} \exp(\Omega(18\epsilon\eta)) \\
&\qquad \exp[16\epsilon\{(\log(10+|\xi|))^{1/2}+1\}\log((\log(10+|\xi|))^{1/2}+1)] \\
&\leq c''_\epsilon(1+|\xi|)^{N_1}(1+|\eta|)^{N_1} \exp(\Omega(18\epsilon\eta)) \exp(48\epsilon\log(10+|\xi|)) \\
&= c''_\epsilon(1+|\xi|)^{N_1+48\epsilon}(1+|\eta|)^{N_1} \exp(\Omega(18\epsilon\eta)).
\end{aligned}$$

On the other hand, there exist c_0 and N_0 such that

(10)

$$\begin{aligned}
\sup_{\substack{|z-\zeta|<r\\ z\in C^n}} |\hat{T}(z)| &= \sup_{|z|<r} |\hat{T}(z+\zeta)| \geq \sup_{|z|\leq M(\frac{1}{2}(\log(\log(10+|\zeta|)))} |\hat{T}(z+\xi)| \\
&\geq \frac{c_0}{(1+|\xi|)^{N_0}}
\end{aligned}$$

by the condition (2).

Applying now to the functions $\hat{\psi}, \hat{T}$ and $\hat{\psi}/\hat{T} = \hat{\varphi}$ Hörmander's lemma with r given by (8) and making use of the estimates (9) and (10), we

obtain
(11)
$$|\hat{\varphi}(\zeta)| \le c_\epsilon(1+|\xi|)^{N_1+48\epsilon+2N_0}(1+|\eta|)^{N_1}\exp(\Omega(18\epsilon\eta))\sup_{|z|<4r}|\hat{\psi}(z+\zeta)|,$$

where c_ϵ is another constant depending on ϵ. But for any integer $l>0$ and all $z=x+iy\in C^n$ with $|z|<4r$, we have, in view of (5),

(12)
$$\begin{aligned}
&|\hat{\psi}(z+\zeta)| \le (1+|z+\zeta|)^{-l}\exp(\Omega(\frac{1}{l}(y+\eta)))w_l(\hat{\psi})\\
&\le (1+|\xi|)^{-l}(1+|z|)^l\exp(\Omega(\frac{2}{l}\eta))\exp(\Omega(\frac{2}{l}y))w_l(\hat{\psi})\\
&\le (1+|\zeta|)^{-l}[1+4M(\frac{1}{2}\log(\log(10+|\xi|)))+4|\eta|]^l\exp(\Omega(\frac{2}{l}\eta))\\
&\qquad \exp(\Omega(\frac{8}{l}M(\frac{1}{2}\log(\log(10+|\xi|)))+\frac{8}{l}|\eta|)w_l(\hat{\psi})\\
&\le (1+|\xi|)^{-l}(1+|\eta|)^l(1+4|\eta|+4(\log(10+|\xi|))^{1/2})^l\exp(\Omega(\frac{2}{l}\eta))\\
&\qquad \exp(\Omega(\frac{16}{l}\eta))\exp[\frac{8}{l}\{\log(10+|\xi|))^{1/2}+1\}\log((\log(10+|\xi|))^{1/2}+1)w_l(\\
&\le c_l(1+|\xi|)^{-l}(1+|\eta|)^l(1+|\eta|)^l(1+|\xi|)^{2/l}\exp(\Omega(\frac{18}{l}\eta))\\
&\qquad \exp(\frac{24}{l}\log(10+|\xi|))w_l(\hat{\psi})\\
&\le c_l'(1+|\xi|)^{-l+\frac{l}{2}+\frac{24}{l}}(1+|\eta|)^{2l}\exp(\Omega(\frac{18}{l}\eta)w_l(\hat{\psi}),
\end{aligned}$$

where c_l and c_l' are constants depending only on l. Consequently from (11) and (12) it follows that

$$\begin{aligned}
w_k(\hat{\varphi}) &= \sup_\zeta(1+|\zeta|)^k\exp(-\Omega(\frac{\eta}{k}))|\hat{\varphi}(\zeta)|\\
&\le c_{\epsilon,l}(1+|\xi|)^{-\frac{l}{2}+\frac{24}{l}+N_1+48\epsilon+2N_0+k}(1+|\eta|)^{k+2l+N_1}\\
&\qquad \exp(\Omega(18\epsilon\eta)+\Omega(\frac{18}{l}\eta)-\Omega(\frac{\eta}{k}))w_l(\hat{\psi})
\end{aligned}$$

For a given k, taking ϵ and l such that $-\frac{l}{2}+\frac{24}{l}+N_1+48\epsilon+2N_0+k<0$ and $\frac{1}{k}-18\epsilon-\frac{18}{l}>0$, we have

$$w_k(\hat{\varphi}) \leq c(1+|\eta|)^{k+N_1+l}\exp(-\Omega(\frac{1}{k_1}\eta))w_l(\hat{\psi}), \tag{13}$$

where $\frac{1}{k_1}=\frac{1}{k}-18\epsilon-\frac{18}{l}$. Since the quantity in (13)

$$\begin{aligned}(1+|\eta|)^{k+N_1+l}&\exp(-\Omega(\frac{1}{k_1}\eta))\\ &\leq (1+|\eta|)^{k+N_1+l}\exp\{(\frac{1}{k_1}|\eta|+1)(1-\log(\frac{1}{k_1}|\eta|+1))\}\end{aligned}$$

is bounded as $|\eta|\to\infty$, we can conclude that

$$w_k(\hat{\varphi}) \leq cw_l(\hat{\psi}) = cw_l(\hat{T}\hat{\varphi})$$

for some c independent of $\hat{\varphi}$. This proves the continuity of the mapping $\hat{T}\hat{\varphi}\to\hat{\varphi}$ and thus completes the proof.

Proof of the theorem 2. We are going to prove the following statement which is stronger than that of the theorem: There exist positive constants B, C, N such that

$$\sup_{|z|\leq M(\frac{1}{2}\log(10+|\xi|))} |\hat{S}(z+\xi)| \geq \frac{c}{(1+|\xi|)^N} \tag{14}$$

for all $\xi\in R^n$ with $|\xi|\geq B$.

Suppose that (14) does not hold. Then there exist a sequence $\{\xi_j\}$ in R^n such that $|\xi_j|\to\infty$ as $j\to\infty$, and

$$\sup_{|z|\leq M(\frac{1}{2}\log(10+|\xi_j|))} |\hat{S}(z+\xi_j)| \leq \frac{1}{(1+|\xi_j|)^j}. \tag{15}$$

Choose $\varphi\in\mathcal{K}_e$ such that $\varphi\geq 0$, $\text{supp}\varphi\subset\overline{B(0,1)}$ and $\hat{\varphi}(0)=1$. By means of φ, we define a sequence $\varphi_j(x)$ of functions as $e^{i<x,\xi_j>}\varphi(x)$. Then φ_j's are in $\mathcal{K}_e$, supported in $\overline{B(0,1)}$ and $\hat{\varphi}_j(\xi_j)=\hat{\varphi}(0)=1$. Since

$S * \mathcal{K}'_e = \mathcal{K}'_e$, there exists a distribution $E \in \mathcal{K}'_e$ such that $S * E = \delta$. Therefore we have

$$\begin{aligned}|\varphi_j(x)| &= |< S * E, \tau_x \check{\varphi}_j >| \\ &= |< E, \check{S} * \tau_x \check{\varphi}_j >| \\ &= |< \hat{E}, \tau_{-x}\widehat{(S * \varphi_j)} >|.\end{aligned}$$

From the continuity of $\hat{E} \in K'_e$, there exist c and k such that

(16)

$$\begin{aligned}|\varphi_j(x)| &\le c w_k(\tau_{-x}\widehat{(S * \varphi_j)}) = c w_k(\hat{S}(\zeta)\hat{\varphi}_j(\zeta)e^{i<x,\zeta>}) \\ &= c \sup_{\zeta=\xi+i\eta\in C^n} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}_j(\zeta)||e^{i<x,\zeta>}| \\ &= c \sup_{|\zeta-\xi_j|\le M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}_j(\zeta)||e^{i<x,\zeta>}| \\ &\quad + c \sup_{|\zeta-\xi_j|> M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}_j(\zeta)||e^{i<x,\zeta>}|\end{aligned}$$

Using the estimate of the form (5) for $\epsilon < \frac{1}{k}$ and $N > k$, the first term in (16) can be estimated as follows;

(17)

$$\begin{aligned}&\sup_{|\zeta-\xi_j|\le M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}_j(\zeta)||e^{i<x,\zeta>}| \\ &\le c_1 \sup_{|\zeta-\xi_j|\le M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)|(1+|\zeta|)^{-N}e^{\Omega(\epsilon\eta)}e^{-<x,\eta>} \\ &\le c_1 \sup_{|\zeta-\xi_j|\le M(\frac{1}{2}\log(10+|\xi_j|))} |\hat{S}(\zeta)|e^{-\Omega(\frac{\eta}{k})+\Omega(\epsilon\eta)+|\eta|} \\ &\le c'_1 \sup_{|\zeta|\le M(\frac{1}{2}\log(10+|\xi_j|))} |\hat{S}(\zeta+\xi_j)| \\ &\le \frac{c'_1}{(1+|\xi_j|)},\end{aligned}$$

where we used that $-\Omega(\frac{\eta}{k}) + \Omega(\epsilon\eta) + |\eta|$ is bounded in R^n and φ_j is supported in the unit ball. In view of the estimate (6) with $\epsilon < \frac{1}{k}$ and the estimate (5) with $\epsilon_1 < \frac{1}{k} - \epsilon$ and N_1 sufficiently large, we estimate the second term in (16) as follows;

(18)

$$\begin{aligned}
&\sup_{|\zeta-\xi_j|>M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}_j(\zeta)||e^{i<x,\zeta>}| \\
&= \sup_{|\zeta-\xi_j|>M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{S}(\zeta)||\hat{\varphi}(\zeta-\xi_j)||e^{i<x,\zeta>}| \\
&\le c_2 \sup_{|\zeta-\xi_j|>M(\frac{1}{2}\log(10+|\xi_j|))} (1+|\zeta|)^{k+N}(1+|\zeta-\xi_j|)^{-N} e^{|\eta|-\Omega(\frac{\eta}{k})+\Omega(\epsilon\eta)+\Omega(\epsilon_1\eta)} \\
&\le c_2 \sup_{|\zeta-\xi_j|>M(\frac{1}{2}(\log(10+|\xi_j|))} (1+|\xi_j|)^{k+N}(1+|\zeta-\xi_j|)^{k+N-N_1} e^{|\eta|-\Omega((\frac{1}{k}-\epsilon-\epsilon_1)\eta} \\
&\le c_2(1+|\xi_j|)^{k+N}(1+M(\frac{1}{2}\log(10+|\xi_j|)))^{k+N-N_1} e^{|\eta|-\Omega((\frac{1}{k}-\epsilon-\epsilon_1)\eta)} \\
&\le c_2'(1+|\xi_j|)^{k+N}(1+|\xi_j|)^{\frac{1}{2}(k+N-N_1)} \\
&\le c_2'(1+|\xi_j|)^{\frac{3}{2}(k+N)-\frac{1}{2}N_1},
\end{aligned}$$

where we used the estimate of the form (5) with $\epsilon_1 < \frac{1}{k} - \epsilon$ and $N_1 > 3(k+N)$, $e^{|\eta|-\Omega((\frac{1}{k}-\epsilon_1-\epsilon)\eta)}$ being bounded in R^n and $M(\frac{1}{2}\log(10+|\xi|)) > \frac{1}{4}(10+|\xi|)^{1/2}$.

Substituting the estimates (17), (18) into (16), we have $|\varphi_j(x)| \le c(1+|\xi_j|)^{-m}$ for some positive integer m and constant c independent on j. But

$$1 = \hat{\varphi}(0) = \hat{\varphi}_j(\xi_j) \le \int_{\overline{B(0,1)}} |\varphi_j(x)|dx \le c(1+|\xi_j|)^{-m} vol(B(0,1))$$

for each $j = 1, 2, \cdots$. This inequality does not hold for $j \to \infty$, which provides a contradiction.

References

1. I. Cioranescu, *Sur les solutions fundamentales d'ordre fini de croissance*, Math. Ann. **211** (1974), 37–46.
2. I. M. Gel'fand and G. E. Shilov, *Generalized functions, Vol II, III*, Academic press, New York, 1968.

3. L. Hörmander, *On the range of convolution operators*, Annals of Math. **76** (1962), 148-170.
4. D. H. Pahk, *Structure theorem and Fourier transform for distributions*, Kyungpook Math. **23** (1965), 351–407.
5. ———, *Hypoelliptic convolution equations in the space $\mathcal{K}'_e$*, Trans. Amer. Math. Soc. **298** (1986), 485–496.
6. S. Sznajder and Z. Zielezny, *Solvability of convolution equations in $\mathcal{K}'_1$*, Proc. Amer. Math. Soc. **57** (1976), 103–106.
7. ———, *Solvability of convolution equations in $\mathcal{K}_p$, $p > 1$*, Pacific J. Math. **68** (1976), 539–544.

Department of Mathematics
Yonsei University
Seoul 120-749, Korea
and
Department of Mathematics
Injae University
Kimhae 621-749, Korea