Comm. Korean Math. Soc. 6(1991), No. 1, pp. 101–107

F–RATIONALITY OVER A QUOTIENT RING OF A C–M LOCAL RING

YONG SU SHIN AND YOUNG HYUN CHO

0. Introduction

Throughout this paper all rings are commutative, Noetherian and characteristic p unless otherwise specified.

In [2], R. Fedder and K. Watanabe proved that if R is a F-rational ring which can be represented as the quotient of a Cohen–Macaulay (for short, C–M) ring, then R is normal and C–M.

In this paper, we shall prove that if R is C–M and if there exists a system of parameters ideal which is tightly closed, then every ideal generated by part of a system of parameters is tightly closed in Theorem 2.2 and if R is an equidimensional ring with $\dim R = d$ which can be represented as the quotient of C–M (respectively, Gorenstein) local ring and if there exists a system of parameters ideal which is tightly closed, then R is C–M (respectively, Gorenstein) and normal in Theorem 3.3. Thus we see that if R is as in Theorem 3.3, then every ideal generated by part of a system of parameters is tightly closed by Theorem 2.2.

1. Preliminaries and Definitions

DEFINITION 1.1. (Hochster–Huneke) Let $I \subseteq R$ be an ideal. If R is a ring with characteristic $p > 0$, we say that $x \in R$ is in the **tight closure**, I^*, of I, if there exists $c \in R^0$ such that for all $e \gg 0$, $cx^{p^e} \in I^{[p^e]}$, where $R^0 = R \setminus \cup\{P : P \text{ is a minimal prime ideal in } R\}$ and $I^{[q]} = (i^q : i \in I)$ when $q = p^e$. If $I = I^*$, we say that I is **tightly closed**.

REMARK 1.2. If R is regular, then $I = I^*$ for all I. (ref., [3], [4], [5], [6]).

Received January 21, 1991.
This was partially supported by the Basic Science Research Institute program, Ministry of Education, 1990.

DEFINITION 1.3. (Hochster–Huneke) R is **weakly F-regular** if every ideal in R is tightly closed. R is **F-regular** if R_p is weakly F-regular for every $p \in \mathrm{Spec}(R)$.

DEFINITION 1.4. (Fedder–Watanabe) R is **F-rational** if every ideal generated by a system of parameters is tightly closed.

Let R be a ring and $I \subset R$ be an ideal. For an R–module M, define

$$\begin{aligned}\Gamma_I(M) &= \{m \in M \mid \mathrm{rad}(\mathrm{Ann}\ m) \supseteq I\} \\ &= \{m \in M \mid \text{There exists a positive integer} \\ &\qquad\qquad n = n(m) \text{ such that } I^n \cdot m = 0\} \\ &\cong \varinjlim \ \mathrm{Hom}_R(R/I^n,\ M).\end{aligned}$$

If $I = (f_1, \ldots, f_r)$, then $\Gamma_I(M) \cong \varinjlim\ \mathrm{Hom}_R(R/(f_1^n, \ldots, f_r^n),\ M)$. (ref. Bemerkung 4.2 in [1].)

DEFINITION 1.5. $\mathrm{H}_I^i(\cdot)$ denotes the right derived functor of $\Gamma_I(\cdot)$ of dimension i. We call $\mathrm{H}_I^i(M)$ the **i–th dimensional local cohomology** of M with respect to I,

$$\text{i.e.,}\quad \mathrm{H}_I^i(M) \cong \varinjlim\ Ext_R^i(R/(f_1^n, \ldots, f_r^n), M)$$

LEMMA 1.6. *Let (R, m) be a C–M local ring, $x_1, \cdots, x_d$ be a system of parameters for R and let $J_i = (x_1, \cdots, x_i)$ for every i. Then* $\mathrm{H}^i_{J_i}(R) = \varinjlim_n R/(x_1^n, \cdots, x_i^n)$.

Proof. We will use induction on $\dim R = d$. Assume that $\dim R = 1$. By Definition 1.5,

$$\mathrm{H}_m^1(R) = \varinjlim \mathrm{Ext}_R^1(R/(x_1^n), R),$$

where x_1 is a system of parameter of R. Since

$$0 \longrightarrow R \xrightarrow{x_1^n} R \longrightarrow R/x_1^n R \longrightarrow 0$$

is exact, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/x_1^n R, R) \longrightarrow \mathrm{Hom}_R(R, R) \xrightarrow{x_1^n} \\ \mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Ext}_R^1(R/x_1^n R, R) \longrightarrow 0$$

and so $0 \longrightarrow R \xrightarrow{x_1^n} R \longrightarrow \mathrm{Ext}_R^1(R/x_1^n R, R) \longrightarrow 0$ is exact. This means that $\mathrm{Ext}_R^1(R/x_1^n R, R) \simeq R/x_1^n R$. Thus

$$\mathrm{H}_m^1(R) = \varinjlim \mathrm{Ext}_R^1(R/x_1^n R, R) = \varinjlim R/x_1^n R.$$

Now assume that $\dim R = d \geq 2$. Let $R/(x_1^n, \cdots, x_i^n) = R_i$.

$$\textbf{Claim}: \mathrm{Ext}_R^j(R_i, R) = \begin{cases} 0, & j \neq i \\ R_i, & i = j \text{ for } j \geq 1. \end{cases}$$

Proof of Claim. For $i = 1$, since $0 \longrightarrow R \xrightarrow{x_1^n} R \longrightarrow R_1 \longrightarrow 0$ is exact, we see that

$$0 \longrightarrow \mathrm{Hom}_R(R_1, R) \longrightarrow \mathrm{Hom}_R(R, R) \xrightarrow{x_1^n} \\ \mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Ext}_R^1(R_1, R) \longrightarrow \mathrm{Ext}_R^1(R, R) = 0$$

is exact. So $0 \longrightarrow R \xrightarrow{x_1^n} R \longrightarrow \mathrm{Ext}_R^1(R_1, R) \longrightarrow 0$ is exact. It follows that $\mathrm{Ext}_R^1(R_1, R) \simeq R_1$ and $0 = \mathrm{Ext}_R^{j-1}(R, R) \longrightarrow \mathrm{Ext}_R^j(R_1, R) \longrightarrow \mathrm{Ext}_R^j(R, R) = 0$ is exact for $j > 1$, i.e., $\mathrm{Ext}_R^j(R_1, R) = 0$ for $j > 1$.

For every $i > 1$, $0 \longrightarrow R_{i-1} \xrightarrow{x_i^n} R_{i-1} \longrightarrow R_i \longrightarrow 0$ is exact. Thus by induction

$$0 = \mathrm{Ext}_R^{i-2}(R_{i-1}, R) \longrightarrow \mathrm{Ext}_R^{i-1}(R_i, R) \longrightarrow \mathrm{Ext}_R^{i-1}(R_{i-1}, R) \xrightarrow{x_i^n} \\ \mathrm{Ext}_R^{i-1}(R_{i-1}, R) \longrightarrow \mathrm{Ext}_R^i(R_i, R) \longrightarrow \mathrm{Ext}_R^i(R_{i-1}, R) = 0$$

is exact. Thus we can obtain an exact sequence,

$$0 \longrightarrow \mathrm{Ext}_R^{i-1}(R_i, R) \longrightarrow \mathrm{Ext}_R^{i-1}(R_{i-1}, R) \xrightarrow{x_i^n} \\ \mathrm{Ext}_R^{i-1}(R_{i-1}, R) \longrightarrow \mathrm{Ext}_R^i(R_i, R) \longrightarrow 0.$$

Hence the following sequence

$$0 \longrightarrow \mathrm{Ext}_R^{i-1}(R_i, R) \longrightarrow R_{i-1} \xrightarrow{x_i^n} R_{i-1} \longrightarrow \mathrm{Ext}_R^i(R_i, R) \longrightarrow 0$$

is exact. So we see that

$$\mathrm{Ext}_R^{i-1}(R_i, R) = 0 \text{ and } \mathrm{Ext}_R^i(R_i, R) = R_i.$$

Similarly, $\mathrm{Ext}_R^j(R_i, R) = 0$ for $j \neq i$. Therefore for every $n \geq 1$ and $i \geq 1$,

$$\mathrm{Ext}_R^i(R/(x_1^n, \cdots, x_i^n), R) \simeq R/(x_1^n, \cdots, x_i^n)$$

whence

$$\mathrm{H}_{J_i}^i(R) = \varinjlim R/(x_1^n, \cdots, x_i^n),$$

as desired.

LEMMA 1.7 ([4] (5.10) COROLLARY). *Let R be a Noetherian ring such that no prime is both minimal and maximal. If every principal ideal of height one is tightly closed, then R is normal.*

2. F-rationality over C–M Local Ring

Let R be a ring of characteristic p. Denote by ${}^eR$, the ring R viewed as an R-module via the e^{th} power of the Frobenius map. Furthermore, for any R-module M, we denote by ${}^eM$ the module $M \otimes_R {}^eR$. Note that if R is reduced and $q = p^e$, there is a natural identification of maps $R \xrightarrow{F^e} {}^eR$ with $R \subset R^{\frac{1}{q}}$ and with $R^q \subset R$ where R^n denotes the ring $\{x^n \mid x \in R\}$ for $n = q$ or $\frac{1}{q}$.

The proof of the following Proposition 2.1 is along the lines of the proof of the Proposition 2.2 in [2].

PROPOSITION 2.1. *Let $x_1, \cdots, x_i$ be a part of some system of parameters for a C–M local ring (R, m) and let $J_i = (x_1, \cdots, x_i)$. If for every $0 \neq \eta \in \mathrm{H}_{J_i}^i(R)$, $\bigcap_{e>0}(0 :_R F^e(\eta)) \subset R \setminus R^0$, then J_i is tightly closed. In particular, if $i = d$, then the converse holds.*

THEOREM 2.2. *Let (R, m) be a C–M local ring. If there exists a system of parameters ideal which is tightly closed, then every ideal generated by part of a system of parameters for R is tightly closed.*

Proof. We first note that R is F-rational by Proposition 2.1 since R is C–M. Let $x_1, \cdots, x_d$ be a system of parameters for R and $J_i = (x_1, \cdots, x_i)$ for every i. In particular, we denote J_{d-1} by J. Assume $0 \neq \eta \in \mathrm{H}_J^{d-1}(R)$ and $\bigcap_{e>0}(0 :_R F^e(\eta)) \not\subset R \setminus R^0$. Let $\bar{f} \in R/(x_1^n, \cdots, x_{d-1}^n)$ represent an element η of $\mathrm{H}_J^{d-1}(R)$. Since $\bigcap_{e>0}(0 :_R F^e(\eta)) \not\subset R \setminus R^0$, there exists an element $c \in R^0$ such that $cf^q \in (x_1^n, \cdots, x_{d-1}^n)^{[q]} \subset (x_1^n, \cdots, x_d^n)^{[q]}$ for all $q = p^e > 0$. Hence $f \in (x_1^n, \cdots, x_{d-1}^n)^* \subset (x_1^n, \cdots, x_d^n)^* = (x_1^n, \cdots, x_d^n)$ since R is F-rational. Thus we can obtain an element $g \in (x_d^n)$ such that $\bar{f} = \bar{g} \in R/(x_1^n, \cdots, x_{d-1}^n)$ represents an element $\eta \in \mathrm{H}_J^{d-1}(R)$. Let $g = a_1 x_d^n$ where $a_1 \in R$. Then $cg^q = c(a_1 x_d^n)^q \in (x_1^n, \cdots, x_{d-1}^n)^{[q]}$ for every $q = p^e > 0$. So $ca_1^q \in (x_1^n, \cdots, x_{d-1}^n)^{[q]} \subset (x_1^n, \cdots, x_d^n)^{[q]}$ for every $q = p^e > 0$, i.e., $a_1 \in (x_1^n, \cdots, x_d^n)^* = (x_1^n, \cdots, x_d^n)$ since R is F-rational. Hence there exist elements $\alpha_1, \cdots, \alpha_{d-1}, a_2 \in R$ such that $a_1 = \alpha_1 x_1^n + \cdots + \alpha_{d-1} x_{d-1}^n + a_2 x_d^n$. Let $h = a_2 x_d^{2n}$. Then $\bar{f} = \bar{g} = \bar{h} = \overline{a_2 x_d^{2n}} \in R/(x_1^n, \cdots, x_{d-1}^n)$. Inductively, we can see that $\bar{f}$ is divided by $\overline{x_d^n}$ in $R/(x_1^n, \cdots, x_{d-1}^n)$ infinitely many times. It follows that $\bar{f} = 0$ in $R/(x_1^n, \cdots, x_{d-1}^n)$, a contradiction. This means that for every $0 \neq \eta \in \mathrm{H}_J^{d-1}(R)$, $\bigcap_{e>0}(0 :_R F^e(\eta)) \subset R \setminus R^0$. Thus $J^* = J$ by Proposition 2.1. By the similar method, we can prove that $J_i^* = J_i$ for every i. Therefore, every ideal generated by part of a system of parameters is tightly closed.

REMARK 2.3. Let (R, m) be a commutative Noetherian local ring of positive prime characteristic p with 1. It is worth noting that if R is F-rational, then every ideal generated by part of some system of parameters for R is tightly closed. And also, by Lemma 1.7, R is normal.

3. A Quotient Ring of a C–M Local Ring

LEMMA 3.1 ([3] LEMMA 3.2). *Let $R = S/I$ where S is a C–M local ring and assume R is equidimensional. Let $\{Q_1, \cdots, Q_n\}$ be the minimal primes over I. Assume that $x_1, \cdots, x_d$ are parameters of R. Then there* *exist elements* $z_1, \cdots, z_h$ *in* I *and* $y_1, \cdots, y_d$ *such that the* y_i *lift* x_i, *the* z'*s and* y'*s together form a regular sequence, and there exists a* $c \notin \cup_{i=1}^{n} Q_i$ *and an integer* $q = p^e$ *such that* $cI^{[q]} \subset (z_1, \cdots, z_h)$, *where* $\mathrm{ht}I = h$.

THEOREM 3.2 ([3] THEOREM 3.3). *Let* $R = S/I$ *be equidimensional, where* S *is a* C–M *local ring. Assume that* $\mathrm{Char}(R) = p > 0$. *Let* $x_1, \cdots, x_n$ *be elements of* R *which are part of a system of parameters. Let* $J = (x_1, \cdots, x_{n-1})R$. *Then* $J :_R x_n \subset J^*$.

THEOREM 3.3. *Let* $R = S/I$ *be equidimensional, where* S *is a* C–M *(respectively, Gorenstein) local ring. If there exists a system of parameters* $x_1, \cdots, x_d$ *in* R *such that the ideal* J *generated by this system of parameters is tightly closed, then* R *is* C–M *(respectively, Gorenstein) and normal. In particular, if* $\dim R$=1, *then* R *is regular.*

Proof. Since S is C–M (respectively, Gorenstein), I will contain an S-sequence $z_1, \cdots, z_h$ of length $\mathrm{ht}I$. Let $\bar{S} = S/(z_1, \cdots, z_h)$ and $\bar{I} = I/(z_1, \cdots, z_h)$. Then $R = S/I \simeq \bar{S}/\bar{I}$. Since S is C–M (respectively, Gorenstein), $\bar{S}$ is also C–M (respectively, Gorenstein). Hence we can assume that $\mathrm{ht}I = 0$. Thus for every $c \in S^0$, the image $\bar{c}$ in R is contained in R^0. By Lemma 3.1, there exist elements $y_1, \cdots, y_d$ in S such that the image $\bar{y}_i$ of y_i in R is equal to x_i for every $i = 1, \cdots, d$ and $y_1, \cdots, y_d$ is an S-sequence. By hypothesis, the ideal $J = (x_1, \cdots, x_d)$ is tightly closed in R. Thus, the ideal generated by $y_1, \cdots, y_d$ is tightly closed in S by Lemma (4.11) (b) in [4]. Hence S is F-rational by Proposition 2.1 since S is C–M. It follows that S is normal by Remark 2.3 and so S is a domain. This means that $I = 0$ since $\mathrm{ht}I = 0$. Thus $R = S$, i.e., R is C–M (respectively, Gorenstein) and normal.

REMARK 3.4. Let R be an equidimensional ring which is the homomorphic image of a Gorenstein local ring with 1. If there exists a system of parameters ideal which is tightly closed, then R is Gorenstein by Theorem 3.3. Hence R is strongly F-regular by Proposition 5.1 in [3] and (3.1) Theorem f) in [5].

References

1. Herzog,J. and Kunz,E., *Der Kanonische Modul eines Cohen-Macaulay-Ring*, Lecture Notes in Math. Vol. **238**, 1971.

2. Fedder,R. and Watanabe,K., *A Characterization of F-regularity in Terms of F-purity*, MSRI Publ. No. 15, Springer-Verlag, 1989, 227–245.
3. ______, *Tight Closure, in Commutative Algebra (Proceeding of a Microprogram Held June 15 - July 2, 1987)*, MSRI Publ. **15**, Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo,1989, 305–324.
4. ______, *Tight Closure, Invariant Theory, and The Briançon-Skoda Theorem*, J. of the Amer. Math. Soc. **3** (1990), 31–116.
5. ______, *Tight Closure and Strongly F-regularity*, Celebration of the contributions of Pierre Samuel to Commutative Algebra.
6. ______, *Tightly Closed Ideals*, Bull. Amer. Math. Soc. **18(1)** (1988), 45–48.

Department of Mathematics
Sung Hwa University
Choongnam 330-600, Korea
and
Department of Mathematics
Seoul National University
Seoul 151-742, Korea