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ON THE HARDY–STEIN IDENTITY

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1. Result

Our motivation is the following theorem appeared at [3 Theorem 2.1].

THEOREM A. *Let $0 < p \leq 2$ and let $0 < \alpha \leq p < \infty$. Then there is a positive constiant $C = C(p,\alpha)$ such that*

$$(1) \qquad \|f\|_p^p \leq C(p,\alpha) \int\int_U |f(z)|^{p-\alpha} |f'(z)|^\alpha (\log \frac{1}{|z|})^{\alpha-1} dxdy$$

for all holomorphic f in U with $f(0) = 0$.

Here $z = x + iy$, U is the unit disc in the complex plane and $\|f\|_p$ denotes the H^p norm of $f(z)$ [1]. If $\alpha = 2$ then (1) follows directly from the Hardy-Stein identity [4] :

$$r\frac{d}{dr} I_p(r,f) = \frac{p^2}{2\pi} \int_0^r \rho d\rho \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2} |f'(\rho e^{i\theta})|^2 d\theta, 0 \leq r < 1,$$

where

$$I_p(r,f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

In the proof of Theorem A, $p \geq \alpha$ was used for the verification that $|f|^{p-\alpha}|f|^\alpha$ is subharmonic. It is natural to consider the right handside integral of (1) for $p < \alpha$ and to ask whether (1) holds at this time also. This note is devoted to this question.

THEOREM. *If $0 < \alpha \leq 2$ and $1 < p < \infty$, then (1) holds for all f holomorphic in U with $f(0) = 0$.*

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2. Proof of theorem

In view of Theorem A, we may assume $p < \alpha$. Fix p and $\alpha : 1 < p < \infty, 1 < \alpha \leq 2$. Let q and β be the conjugate exponents of p and α, respectively. Let f be holomorphic in U with $f(0) = 0$ and let $g \in H^q$. Consider

$$(*) = \int\int_{rU} h(z)|g'(z)|^\beta (\log \frac{r}{|z|})^{\beta-1} dxdy,$$

where $h(z) = |f(z)|^{(\alpha-p)(\beta-1)}$. If we denote $M_r h$ the radial maximal function of $h_r(z) = h(rz) : M_r h(\theta) = \sup_{o \leq \rho < 1} |h(r\rho e^{i\theta})|$, the integral $(*)$ is majorized by

$$\int_0^{2\pi} M_r h(\theta) \int_0^r |g'(\rho e^{i\theta})|^\beta (\log \frac{r}{\rho})^{\beta-1} \rho d\rho d\theta.$$

Applying Hölder's inequality, this is at most

$$(2) \qquad \left[\int_0^{2\pi} M_r h(\theta)^{q/(q-\beta)} d\theta\right]^{q-\beta/q} X$$
$$\left[\int_0^{2\pi} \{\int_0^r |g'(\rho e^{i\theta})|^\beta (\log \frac{r}{\rho})^{\beta-1} \rho d\rho\}^{q/\beta} d\theta\right]^{\beta/q}.$$

Here the first factor is, by the complex maximal theorem [1 Theorem 1.9], at most a constant times $\|f_r\|_p^{p+\beta-\beta p}$. While the second factor of (2) is, after changing variables, at most a constant times

$$\left[\int_0^{2\pi} \left\{\int_0^1 |g_r'(\rho e^{i\theta})|^\beta (1-\rho)^{\beta-1} d\rho\right\}^{q/\beta} d\theta\right]^{\beta/q},$$

which is magjorized by $C(p, \beta)\ \|g_r\|_q^\beta$ [2]. Hence we can conclude

$$(3) \qquad (*) \leq C(p, \alpha) \|f_r\|_p^{p+\beta-\beta p} \|g_r\|_q^\beta.$$

Now we will use a duality argument to prove (1). If follows easily from the Green's theorem that

$$\int_0^{2\pi} f\bar{g}(re^{i\theta}) d\theta = \int\int_{rU} f'\bar{g}'(z) \log \frac{r}{|z|} dxdy.$$

Thus by Hölder's inequality,

$$\left|\int_0^{2\pi} f\bar{g}(re^{i\theta})d\theta)\right| \leq (*)^{1/\beta}\ X$$

$$\left[\int\int_{rU} |f(z)|^{p-\alpha}|f'(z)|^{\alpha}(\log\frac{r}{|z|})dxdy\right]^{1/\alpha}.$$

Therefore, applying (3), we obtain

$$\left|\int_0^{2\pi} f\bar{g}(re^{i\theta})d\theta\right| \leq C(p,\alpha)\|f_r|_p^{\delta}\|g_r\|_q\ X \tag{4}$$

$$\left[\int\int_{rU} |f(z)|^{p-\alpha}|f'(z)|^{\alpha}(\log\frac{r}{|z|})^{\alpha-1}dxdy\right]^{1/\alpha},$$

$\delta = -1 - p + p/\beta$. Since (4) holds for any $g(z) \in H^q(U)$, we conclude that

$$\|f_r\|_p^p \leq C(p,\alpha)\int\int_{rU} |f(z)|^{p-\alpha}|f'(z)|^{\alpha}(\log\frac{r}{|z|})^{\alpha-1}dxdy.$$

Letting $r \to 1$, we arrive at the desired (1).

If $0 < \alpha \leq 1$ then (1) follows from what we have just proved: apply Hölder's inequality to the Hardy-Stein identity, then use (1) for $1 < \alpha \leq 2$.

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