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THE GROUP OF BOUNDED ELEMENTS IN A LIE GROUP

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Let G be a real analytic group, $\mathcal{A}$ an arbitrary, not necessarily connected subgroup of $Aut(G)$. By $B(G,\mathcal{A})$, we mean the group of all elements $x \in G$ such that $\mathcal{A}x$ is relatively compact.

In [1] it is shown that $B(G,\mathcal{A})$ is closed when S is faithfully representable, and in [2] the result is is generalized to the case when $R \cap S$ is finite where R, S denote the solvable radical and a Levi factor of G, respectively.

The purpose of this note is to prove the following therorem, which generalizes the above result.

THEOREM. *Let G be an analytic group. Then $B(G,\mathcal{A})$ is closed for every subgroup $\mathcal{A}$ of $Aut(G)$.*

In proving the theroem we shall use the following two propositions in [2].

PROPOSITION 1. *Let G be an analytic group such that $R \cap S$ is finite, where R, S denote the solvable radical and a Levi factor of G, respectively. Then $B(G,\mathcal{A})$ is closed.*

PROPOSITION 2. *Let $\pi : G' \longrightarrow G$ be a covering homomorphism of semi-simple analytic groups. Let $\mathcal{A}'$ be an arbitrary subgroup of $Aut(G')$. Then, for each $x \in G'$, $\mathcal{A}'x$ is relatively compact if and only if $\pi(\mathcal{A}'x)$ is relatively compact.*

Let G be a given analytic group and let $\phi : \tilde{G} \longrightarrow G$ be a universal covering group of G. Let $\tilde{G} = \tilde{R}\tilde{S}$ be a Levi decomposition of $\tilde{G}$. Then $G = RS$ is a Levi decomposition of G, where $R = \phi(\tilde{S})$, $S = \phi(\tilde{S})$. Let $G' = R \times_\sigma \tilde{S}$ be a semidirect product, where $\sigma : \tilde{S} \longrightarrow Aut(R)$ is a continuous homomorphism defined by $\sigma(\tilde{s})(r) = \phi(\tilde{s})r(\phi(\tilde{s}))^{-1}$, $r \in R$, $\tilde{s} \in \tilde{S}$.

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For $\alpha \in Aut(G)$, choose $\tilde{\alpha} \in Aut(\tilde{G})$ such that $\phi \circ \tilde{\alpha} = \alpha \circ \phi$ and define $\alpha' : G' \longrightarrow G'$ by $\alpha'(r, \tilde{s}) = (\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s}))), \ p_2(\tilde{\alpha}(\tilde{s})))$, where $p_1 : \tilde{G} \longrightarrow \tilde{R}$ and $p_2 : \tilde{G} \longrightarrow \tilde{S}$ are the projections to the first and second factor of $\tilde{G} = \tilde{R}\tilde{S}$, respectively *i.e.*, $p_1(\tilde{r}\tilde{s}) = \tilde{r}$ and $p_2(\tilde{r}\tilde{s}) = \tilde{s}$, for $\tilde{s} \in \tilde{R}$, $\tilde{s} \in \tilde{S}$.

LEMMA 3. *Let G be an analytic group and let $\pi : G' \longrightarrow G$ be a mapping defined by $\pi(r, \tilde{s}) = r\phi(\tilde{s})$. Then π is a covering homomorphism such that $\pi \circ \alpha' = \alpha \circ \pi$ and $\alpha' \in Aut(G')$ for each $\alpha \in Aut(G)$.*

Proof. It is clear that π is a continuous open epimorphism with discrete kernel, and hence it is a covring homomorphism. By definition, it is clear that α' is continuous and direct calculation shows that $\pi \circ \alpha' = \alpha \circ \pi$. Therefore we have $(\alpha\beta)' = \alpha' \circ \beta'$ for $\alpha, \beta \in Aut(G)$ and $(id_G)' = id_G$, by the unique lifting property. For $x \in G'$, consider the mapping $f : G' \longrightarrow G'$ defined by $f(y) = (\alpha'(x))^{-1}\alpha'(xy)$, $y \in G'$. The unique lifting property shows that $f = \alpha'$, proving the Lemma.

PROPOSITION 4. *Let $\mathcal{A}' = \{\alpha' : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is an arbitrary subgroup of $Aut(G)$, and let $x = (r, \tilde{s}) \in G'$. Then $\mathcal{A}'x$ is relatively compact if and only if $\pi(\mathcal{A}'x)$ is relatively compact, where π is the covering homorphism defined in Lemma 3.*

Proof. The only if part is trivial. To prove the converse let $\iota : \tilde{S} \longrightarrow \tilde{G}$ be the inclusion and let $\psi : G \longrightarrow G/R$ be the canonical epimorphism. Then $\psi \circ \phi \circ \iota : \tilde{s} \longrightarrow G/R$ is a covering homomorphism and a simple calculation shows that $\{p_2 \circ \tilde{\alpha} \circ \iota : \alpha \in \mathcal{A}\}$ is a subgroup of $Aut(\tilde{S})$. Since $\{(\psi \circ \phi \circ \iota)(p_2 \circ \tilde{\alpha} \circ \iota(\tilde{s})) : \alpha \in \mathcal{A}\} = \psi(\pi(\mathcal{A}'x))$ is relatively compact by assumption, we see that $\{p_2 \circ \tilde{\alpha} \circ \iota(\tilde{s}) : \alpha \in \mathcal{A}\}$ is relatively compact by Proposition 2. Therefore the relative compactness of $\{\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s})) : \alpha \in \mathcal{A}\}$ follows from the relative compactness of $\{\alpha(r)\phi(p_1(\tilde{\alpha}(\tilde{s}))\phi(p_2 \circ \tilde{\alpha}(\tilde{s})) : \alpha \in \mathcal{A}\}$. This shows that $\mathcal{A}'$x is relatively compact, which completes the proof.

THEOREM 5. *$B(G, \mathcal{A})$ is closed for every analytic group G and a subgroup $\mathcal{A}$ of $Aut(G)$.*

Proof. Proposition 4 shows that $B(G', \mathcal{A}') = \pi^{-1}(B(G, \mathcal{A}))$. Since $BG', \mathcal{A}')$ is closed by Proposition 1, it follows that $B(G, \mathcal{A})$ is closed.

References

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