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NORMALIZING EXTENSIONS OF RINGS AND SUBNORMALIZING EXTENSIONS OF RINGS

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Let S be a ring with an identity element and let R be a subring of S, sharing the same identity element of S, and S be finitely generated as an R-module by elements $a_1, \cdots, a_n$ with $a_1R = Ra_1$. Then S is called a normalizing extension of R. The relationahips between the prime ideals of those two rings has been extensively studied. This relationship is very similar to the relationship of Krull, and of Cohen and Seidenberg, for integral extensions of a commutative ring. This suggests that a similar relationship could exist between the prime ideals of R and those of any ring T with $R \subset T \subset S$, such a ring T being termed an intermediate normalizing extension of R.

A ring extension $S \supset R$ is called a finite subnormalizing extension in case there are finite number of elements $a_1, \cdots, a_n$ of S such that $S = \sum_{i=1}^{n} Ra_i$ and $\sum_{i=1}^{j} Ra_i = \sum_{i=1}^{j} a_iR (j = 1, 2, \cdots, n)$ [4]. This is a generalization of normalizing extension.

In this paper we consider the radical structure between R and T where T is an intermediate normalizing extension of R. Also we provide a counter example which shows that the notion of finite subnormalizing extension is weaker than that of finite intermediate normalizing extension. Throughout this paper all rings have an identity element, ring extensions are unitary, and all modules are unitary.

Let $J(R)$ denote the Jacobson radical of R and $P(R)$ denote the prime radical of R.

DEFINITION 1. The sum of all locally nilpotent ideals of R is called the Levitzki radical of R and denoted by $L(R)$ [1].

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DEFINITION 2. The intersection of all the maximal ideals of R is called the Brown-McCoy radical of R and denoted by $G(R)$ [1].

It is well-known that $P(R) \subset L(R) \subset J(R) \subset G(R)$.

THEOREM 3. *If T is an intermediate normalizing extension of R, then $J(R) = J(T) \cap R$ and $P(R) = P(T) \cap R$ [2].*

THEOREM 4. *Let S be a normalizing extension with $R \subset T \subset S$ where T is an intermediate normalizing extension of R. Then $L(T) \cap R = L(R)$.*

Proof. We first prove that if P and J are linked pair (in the sense of [2 Definition 2.15]), then P is a prime ideal in R such that R/P is Levitzki semisimple if and only if T/J is Levitzki semisimple. To prove this we may assume that T is prime and Levitzki semisimple. By [2 Theorem 2.13], 0 is a finite intersection of prime ideals with mutually isomorphic factor rings of R. If $L(R/P) \neq 0$, $J(R/P)[x,y] = L(R/P)[x,y] \neq 0$ where x and y are noncommuting indeterminiates [3]. By theorem 3, $J(T[x,y]) \neq 0$ and $J(T[x,y]) = L(T)[x,y] \neq 0$. Therefore $L(T) \neq 0$. This contradictsd to $L(T) = 0$. Thus $L(R/P) = 0$.

Conversely assume that T is a prime ring and $L(R/P) = 0$. If $L(T) \neq 0$, then by [2], $R \cap L(T) \not\subseteq P$. $[(R \cap L(T)) + P]/P$ is a locally nilpotent ideal of R/P. This contradicts to $L(R/P) = 0$. $L(T) = \cap\{J | J$ is a prime ideal in T and T/J is Levitzki semi-simple $\}$ [1].

Let J be a prime ideal in T such that T/J is Levitzki semi-simple. $J \cap R$ is a finite intersection of prime ideals such that factor rings are Levitzki semi-simple. Thus $L(T) \cap R \supseteq L(R)$. Since $L(T) \cap R$ is a locally nilpotent ideal of R, $L(T) \cap R \subseteq L(R)$. This completes the proof.

THEOREM 5. *Let S be a normalizing extension of R with $R \subset T \subset S$ and T be any intermediate normalizing extension of R. Then $G(T) \cap R = G(R)$.*

Proof. . Let J be a maximal ideal in T. $J \cap R$ is a finite intersection of maximal ideals of R by [2, Theorem 4.5]. Thus $G(T) \cap R \supseteq G(R)$. Let P be a maximal ideal of R. There is a maximal ideal J in T such that P is minimal over $J \cap R$ [2, Theorem 4.1]. Therefore $G(T) \cap R \subset G(R)$. Thus $G(R) = G(T) \cap R$. This completes the proof.

Let R be a ring and I be an ideal of R. Let $P(I)$ be the intersection of all prime ideals in R containing I and $J(I)$ be the intersection of all left primitive ideals of R containing I. Similarly let $G(I)$ be the intersection of all maximal ideals in R containing I.

DEFINITION 6. Let R be a ring.

(1) R is said to be a Jacobson ring if for every ideal I of R, $P(I) = J(I)$.
(2) R is said to be a Brown-McCoy ring if for every ideal I of R, $P(I) = G(I)$.

THEOREM 7. *Let R be a ring and let S be a normalizing extension of R and T be an intermediate normalizing extension of R. Then R is a Jacobson ring if and only if T is a Jacobson ring.*

Proof. Let R be a Jacobson ring. We first show that if T is a prime ring, then $J(T) = 0$ where $J(T)$ is the Jacobson radical of T. There are prime ideals $P_1 \cdots P_t$ of R such that $P_1 \cap \cdots \cap P_t = 0$ [2]. Since R is a Jacobson ring, $P(R) = J(R) = 0$. Suppose that $J(T) \neq 0$. Then $J(T) \cap R \neq 0$ [2, 2.13]. Thus $J(R) = J(T) \cap R \neq 0$. This contradicts to the fact $J(R) = 0$. Thus $J(T) = 0$. This proves that if R is a Jacobson ring, then T is a Jacobson ring. Conversely assume that R is a prime ring. Let J be a prime ideal of T which is maximal with respect to $J \cap R = 0$. $J \cap R = 0$ is a finite intersection of prime ideals in R. Since T is a Jacobson ring, J is an intersection of primitive ideals, say $J = \cap\{J_i | i \in I\}$ where I is an index set. Fork each J_i, there exists primitive ideals P_{ij} such that $J_i \cap R = \cap_{j \in K_i} P_{ij}$. Thus $\cap\{P_{ij} | i \in I, j \in K_i\} = \cap\{J_i | i \in I\} \cap R = J \cap R = 0$ where K_i is an index set. Hence R is semisimple. This completes the proof.

THEOREM 8. *Let R be a ring. Then we have the following : R is a Brown McCoy ring if and only if T is a Brown McCoy ring where T is an intermediate normalizing extension of R.*

Proof. Let T be a Brown-McCoy ring. We may assume that R is a prime ring. Let J be an ideal of T which is maximal with respect to $J \cap R = 0$. J is a prime ideal in T. Since T is a Brown-McCoy ring, J is an intersection of maximal ideals, say $J = \cap\{J_i | i \in I\}$ where I is an index set. For each J_i, there are maximal ideals P_{ij} such that

$J_i \cap R = \cap\{P_{ij} | j \in K_i\}$ where K_i is an index set. Thus $\cap\{P_{ij} | i \in I, j \in K_i\} = \cap\{J_i | i \in I\} \cap R = J \cap R = 0$. Therefore R is Brown-McCoy semisimple. Conversely assume that R is Brown-McCoy ring. We will show that if T is a prime ring, then $G(T) = 0$ where $G(T)$ is the Brown McCoy radical of T. There are prime ideals $P_1, \cdots, P_t$ of R such that $P_1 \cap \cdots \cap P_t = 0$ where $P_1, \cdots, P_t$ are linked to 0 [2, Definition 2.15]. Since R is Brown - McCoy, $G(R) = P(R) = 0$. Suppose $G(T) \neq 0$. Then $G(T) \cap R \neq 0$ by [2, 3.12]. $G(R) = G(T) \cap R \neq 0$. This contradicts to that $G(R) = 0$. Thus $G(T) = 0$. This proves that if R is a Brown-McCoy ring, then T is a Brown-McCoy ring.

We end this paper with an example which shows that the notion of finite subnormalizing extension is much weaker than that of finite normalizing extension.

EXAMPLE [5]. . Let $S = M_2(Q), R = \begin{bmatrix} Q & Q \\ O & Q \end{bmatrix}$ $a_1 = 1_s$, $a_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ where Q is the rational number field.

S is a subnormalizing extension of R which is not normalizing extension. $J(R) = P(R) = \begin{bmatrix} O & Q \\ O & O \end{bmatrix} = G(R)$. But $J(S) = P(S) = G(S) = 0$.

The similar results for Theorems 3, 4, 5 are not true for this case.

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