Comm. Korean Math. Soc. 6(1991), No. 1, pp. 73–87

A FOCAL COMPARISON THEOREM FOR NULL GEODESICS

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1. Introduction

Let (M_i, g_i) be arbitrary Lorentzian manifolds of dim $n_i \geq 3$ and let $\beta_i : [0, b] \to M_i$ be null geodesic segments perpendicular at $\beta_i(0)$ to the spacelike submanifolds K_i of dim $k_i \geq 0$, i = 1,2.

S. Kim in [6,8] extended the Lorentzian Rauch comparison theorem for cojugate points given by J. K. Beem and P. E. Ehrlich in [2,3] to K-focal points under suitable curvature and non-focal point condtions, and under the dimenension condition that $n_1 \leq n_2$. Especially, F. Warner in [9] obtained the focal Rauch comparison theorem on Riemannian manifolds with $n_1 > n_2$, k_1 arbitrary, and $k_2 = n_2 - 2$.

In §3 we give a formal setup of curvaturelike tensor classes and their associated C^2 K-Jacobi classes along null geodesics as in the timelike case, cf.[7].

In §4 we show that the focal Rauch comparison theorem for null geodesics is still available for Warner's dimension conditions.

2. Preliminaries

Let (M,g) be an arbitrary Lorentzian manifold of dim M $= n \geq 3$ and let β : [0,b] $\to$ (M,g) be a null geodesic segment perpendicular at $\beta(0) = p$ to the spacelike submanifold K of dim $k, 0 \leq k \leq n-2$, which has the positive definite pullback metric. We recall some notations and definitions for null geodesics, cf.[5,6,8].

One considers an **R**-vector space $V^{\perp}(\beta, K)$ of continuous piecewise smooth vector fields Y along the geodesic β perpendicular to β' with $Y(0) \in T_pK$, and let $V_0^{\perp}(\beta, K) = \{Y \in V^{\perp}(\beta, K) | Y(b) = 0\}$. Then

Received November 15,1990.

This work is supported by the Basic Science Research Institute Program, Ministry of Education, 1988 - 1989 .

we may define K-Jacobi fields and K-focal points along β, and the Lorentzian submanifold index form $I_{(b,K)} : V^{\perp}(\beta, K) \times V^{\perp}(\beta, K) \longrightarrow \mathbf{R}$ given by, for $X, Y \in V^{\perp}(\beta, K)$

$$I_{(b,K)}(X,Y) = g(S_{\beta'(0)}X(0), Y(0)) - \int_0^b [g(X', Y') - g(R(X, \beta')\beta', Y)]dt$$

where $S_{\beta'(0)}$ is the second fundamental operator on T_pK and R is the curvature tensor field with respect to the Levi-Civita connection ∇ on (M,g).

However, we could not characterize K-Jacobi fields in $V_0^{\perp}(\beta, K)$ on an infinite dimensional subspace $\Omega = \{Y = f\beta' | f : [0, b] \rightarrow \mathbf{R}$ with $f(0) = 0\}$ of non-Jacobi fields, cf.[5,6,8].

Thus we construct a quotient bundle

$$G(\beta) = \bigcup_{t \in [0,b]} G(\beta(t)) = \bigcup_{t \in [0.b]} (\beta'(t))^{\perp}/[\beta'(t)]$$

where

$[\beta'(t)] = \{\lambda\beta'(t) \mid \lambda \in \mathbf{R}\}$ and

$(\beta'(t))^{\perp} = \{v \in T_{\beta(t)}M \mid g(v, \beta'(t)) = 0\}$,

and a natural projection $\pi : (\beta'(t))^{\perp} \rightarrow G(\beta(t))$. Then we may define some $\mathbf{R}$-vector spaces as follows:

$\chi(\beta) = \{V \mid V : [0, b] \rightarrow G(\beta)$ is a piecewise smooth section$\}$,

$\chi_0(\beta) = \{V \in \chi(\beta) \mid V(0) = [\beta'(0)], V(b) = [\beta'(b)]\}$,

$\chi(\beta, K) = \{V \in \chi(\beta) \mid V = \pi(X)$ for some X $\in V^{\perp}(\beta, K)\}$,

$\chi_0(\beta, K) = \{V \in \chi(\beta, K) \mid V(b) = [\beta'(b)]\}$.

Moreover,

$$\pi |_{V^{\perp}(\beta,K)}: V^{\perp}(\beta, K) \longrightarrow \chi(\beta, K)$$

is defined as the restriction of the above projection map.

On $\chi(\beta, K)$ we may define a quotient metric $\bar{g}$, the covariant derivative operator $\bar{\nabla}$, the second fundamental tensor $\bar{S}_{\beta'(0)}$, the curvature tensor field $\bar{R}$, a Jacobi class, a K-quotient focal point, and the K-quotient index form as follows; cf.[5,7].

For any V,W $\in \chi(\beta, K)$ with $\pi(X) = V, \pi(Y) = W$ for X,Y $\in V^{\perp}(\beta, K)$,

$\bar{g}(V, W) = g(X, Y)$,

$\bar{\nabla}_{\beta'(t)}V(t) = V'(t) = \pi(\nabla_{\beta'}X(t))$,

$\bar{S}_{\beta'(0)}V(0) = \pi(S_{\beta'(0)}X(0)) = S_{\beta'(0)}X(0) + [\beta'(0)]$,

$\bar{R}(V, \beta')\beta' = \pi(R(X, \beta')\beta')$.

V is called a *K-Jacobi class* on $\chi(\beta, K)$ if V satisfies

(1)$V(0) \in \pi(T_pK)$

(2)$V'(0) + \bar{S}_{\beta'(0)}V(0) \in \pi((T_pK)^{\perp})$

(3)$V'' + \bar{R}(V, \beta')\beta' = [\beta']$

$t_0 \in (0, b]$ is called a *K-quotient focal point* if there exists a nontrivial K-Jacobi class V $\in \chi(\beta, K)$ with $V(t_0) = [\beta'(t_0)]$.

The *K-quotient index form* $\bar{I}_{(b,K)} : \chi(\beta, K) \times \chi(\beta, K) \to \mathbf{R}$ is given by

$$\bar{I}_{(b,K)}(V, W) = \bar{g}(\bar{S}_{\beta'(0)}V(0), W(0)) - \int_0^b [\bar{g}(V', W') - \bar{g}(\bar{R}(V, \beta')\beta', W)]dt.$$

Notice that $\bar{I}_{(b,K)}(V, W) = I_{(b,K)}(X, Y)$ for $X, Y \in V^{\perp}(\beta, K)$ with $\pi(X) = V, \pi(Y) = W$. Therefore, we now characterize the K-Jacobi classes among the piecewise smooth sections in $\chi(\beta, K)$ as in [6,8].

Let (M,g) be a Lorentzian manifold and $\beta : [0, b] \to (M, g)$ be a null geodesic segment perpendicular at $\beta(0)$ to the spacelike submanifold K of dimension k, $0 \leq k \leq n-2$. Then we may choose the spacelike orthonormal vectors as in [6] $e_1, e_2, \ldots\ldots, e_{n-2} \in T_pM$ such that for a null vector n with $g(\beta'(0), n) = -1$ and $n \in (T_pK)^{\perp}$:

$$e_i \in T_pK,\ e_i \perp \operatorname{span}\{\beta'(0), n\} \text{ for } i = 1, 2, \ldots, k$$

and

$$e_j \in (T_pK)^{\perp},\ e_j \perp \operatorname{span}\{\beta'(0), n\} \text{ for } i = k+1, k+2, \ldots, n-2.$$

Inspired by W.Ambrose's idea in [1], we constructed (n-2) K-Jacobi fields $J_1, J_2, \ldots\ldots, J_{n-2}$ such that

$$J_i(0) = e_i, J_i'(0) = -S_{\beta'(0)}e_i \text{ for } i = 1, 2, \ldots, k$$

and

$$J_j(0) = 0, J_j'(0) = e_j \text{ for } i = k+1, k+2, ..., n-2.$$

Then we may prove that $\{J_i\}_{i=1,2,...,n-2}$ are spacelike K-Jacobi fields in $V^{\perp}(\beta, K)$, and $\{J_i\}_{i=1,2,...,n-2}$ project to linearly independent K-Jacobi classes $\{V_i\}_{i=1,2,...,n-2}$ with $V_i = \pi(J_i)$ for $J_i \in V^{\perp}(\beta, K)$, $i = 1, 2, ..., n-2$. More precisely, for any $V \in \chi(\beta, K)$,

$$V = \sum_{i=1}^{n-2} f_i(t) V_i$$

for some piecewise smooth functions f_i on (0,b].

Thus, we obtain the maximality of K-Jacobi classes adapted to the quotient bundle setting, cf.[5,6,8].

THEOREM 2.1. *Let $\beta : [0, b] \to (M, g)$ be a null geodesic segment perpendicular at $\beta(0)$ to the spacelike submanifold K with no K-quotient focal points and let $W \in \chi(\beta, K)$. Then there exists a unique K-Jacobi class $V \in \chi(\beta, K)$ with V(b) = W(b). Further,*

$$\bar{I}_{(b,K)}(W, W) \leq \bar{I}_{(b,K)}(V, V)$$

and equality holds iff $V = W$.

COROLLARY 2.2. *Suppose $\beta : [0, b] \longrightarrow (M, g)$ has no K-quotient focal points. Then K-quotient index form $I_{(b,K)} : \chi(\beta, K) \times \chi(\beta, K) \longrightarrow$ **R** is negative definite.*

Now, let (M_i, g_i) be arbitrary Lorentzian manifolds with $dim M_i = n_i \geq 3, n_1 \leq n_2$, and let $\beta_i : [0, b] \longrightarrow M_i$ be null geodesic segments perpendicular at $\beta_i(0)$ to the spacelike submanifolds K_i of dim $k_i, 0 \leq k_i \leq n_i - 2$, for i = 1,2. Construct a linear isometry using parallel translation

$$\bar{\phi} : \chi(\beta_1) \longrightarrow \chi(\beta_2)$$

by $(\bar{\phi}V)(t) = (\phi_t X)(t) + [\beta_2'(t)]$ for any $V \in \chi(\beta_1)$ with $\pi(X) = V$ for $X \in V^{\perp}(\beta_1)$ where $\phi_t : T_{\beta_1(t)}M_1 \to T_{\beta_2(t)}M_2$ is a linear isometry as in the timelike case, cf.[5,6]. Further,

$$(\bar{\phi}V)'(t) = (\bar{\phi}V')(t)$$

for all $t \in [0, b]$.

Of course, the map $\bar{\phi}$ constructed above will not map $\chi(\beta_1, K_1)$ into $\chi(\beta_2, K_2)$ since we have placed no restrictions on $T_{\beta_1(0)}K_1, T_{\beta_2(0)}K_2$, and a linear isometry $j : T_{\beta_1(0)}M_1 \to T_{\beta_2(0)}M_2$ with $j(\beta_1'(0)) = \beta_2'(0)$.

If $dimK_1 \leq dimK_2$ and if we construct j so that $j(T_{\beta_1(0)}K_1) \subset T_{\beta_2(0)}K_2$ by appropriate choices of orthonormal basis, then we have

$$\bar{\phi}(\chi(\beta_1, K_1)) \subset \chi(\beta_2, K_2).$$

But if $dimK_1 > dimK_2$, no such j can be found. Nonetheless, given a fixed $V \in \chi(\beta_1, K_1)$ with $V(0) \neq [\beta_1'(0)]$ we may construct

$$\bar{\phi} : \chi(\beta_1) \longrightarrow \chi(\beta_2)$$

so that $\bar{\phi}V \in \chi(\beta_2, K_2)$ as follows:

Given $V \in \chi(\beta_1, K_1)$, choose a spacelike vector field X such that $\pi(X) = V$. Let $\beta_1'(0)$, $n, e_1, e_2, \ldots\ldots, e_{n_1-2}$ be a pseudo-orthonormal basis of $T_{\beta_1(0)}M_1$, where

$$e_1 = X(0)/(g_1(X(0), X(0))^{1/2}.$$

Choose orthonormal vectors $\tilde{e}_1, \tilde{e}_2, \ldots\ldots, \tilde{e}_{n_1-2}$ in $(\beta_2(0))^{\perp}$ such that $\tilde{e}_1 \in T_{\beta_2(0)}K_2$. Then if $j : T_{\beta_1(0)}M_1 \longrightarrow T_{\beta_2(0)}M_2$ is defined by setting $j(\beta_1'(0)) = \beta_2'(0), j(n) = \tilde{n}$, and $j(e_i) = \tilde{e}_i$ with $g_2(n, \beta_2'(0)) = g_2(n, \tilde{e}_i) = 0$ for $1 \leq i \leq n_1 - 2$, then $(\phi X)(0) = (g_1(X(0), X(0))^{1/2}\tilde{e}_1 \in T_{\beta_2(0)}K_2$, so that $\phi X \in V^{\perp}(\beta_2, K_2)$. Therefore, $\bar{\phi}V \in \chi(\beta_2, K_2)$.

To avoid repetition of boring statements we notate as follows:

Condition($\bar{\mathbf{R}}$) : the maximum value of $\bar{g}_1(\bar{R}_1(\bar{u}, \beta_1'(t))\beta_1'(t), \bar{u})$ for $\bar{u} \in (\beta'(t))^{\perp}/[\beta'(t)]$ with $\bar{g}_1(\bar{u}, \bar{u}) = 1$ is less than or equal to the minimum value of $\bar{g}_2(\bar{R}_2(\bar{v}, \beta_2'(t))\beta_2'(t), \bar{v})$ for $\bar{v} \in (\beta_2'(t))^{\perp}/[\beta_2'(t)]$ with $\bar{g}_2(\bar{v}, \bar{v}) = 1$ for each $t \in [0, b]$.

Condition($\bar{\mathbf{S}}$) : the maximum value of $\bar{g}_1(S_{\beta_1'(0)}\bar{u}, \bar{u})$ for $\bar{u} \in \pi(T_{\beta_1(0)}K_1)$ with $\bar{g}_1(\bar{u}, \bar{u}) = 1$ is less than or equal to the minimum value of $\bar{g}_2(S_{\beta_2'(0)}\bar{v}, \bar{v})$ for $\bar{v} \in \pi(T_{\beta_2(0)}K_2)$ with $\bar{g}_2(\bar{v}, \bar{v}) = 1$.

Now we use Theorem 2.1 to extend the index comparison theorem given by J. K. Beem and P. E. Ehrlich as in [2,3] to the K-quotient index form $\bar{I}_{(b.K)}$ as in the timelike case, cf.[6,7].

THEOREM 2.3. *Let* (M_i, g_i) *be arbitrary Lorentzian manifolds with* $dim M_1 = n_1 \leq n_2 = dim M_2$ *and let* $\beta_i : [0, b] \longrightarrow M_i$ *be null geodesics perpendicular to the spacelike submanifolds* K_i *with* $dim k_i > 0$, $i = 1,2$. *Suppose the Conditions* ($\bar{\mathbf{R}}$) *and* ($\bar{\mathbf{S}}$) *hold. Then, given* $V \in \chi(\beta_1, K_1)$, *constructing* $\bar{\phi}$ *as above so that* $\bar{\phi}V \in \chi(\beta_2, K_2)$,

$$\bar{I}_{(b,K_1)}(V, V) \leq \bar{I}_{(b,K_2)}(\phi V, \phi V).$$

Further, for dimensions that $n_1 \leq n_2$ we may extend the null Rauch comparison theorem for conjugate points given by J. K. Beem and P. E. Ehrlich in [2,3] to K-quotient focal points, cf.[8].

THEOREM 2.4. *Let* (M_i, g_i) *be arbitrary Lorentzian manifolds with* $dim M_1 = n_1 \leq n_2 = dim M_2$ *and let* $\beta_i : [0, b] \longrightarrow M_i$ *be null geodesics perpendicular to the spacelike submanifolds* K_i, $i = 1,2$. *For each* $t \in [0, b]$, *assume the conditions* ($\bar{\mathbf{R}}$) *and* ($\bar{\mathbf{S}}$) *hold and let* $\bar{X} \in \chi(\beta_1, K_1), \bar{Y} \in \chi(\beta_2, K_2)$ *be* K_1- *and* K_2 *-quotient Jacobi classes respectively such that* $\bar{g}_1(\bar{X}(0), \bar{X}(0)) = \bar{g}_2(\bar{Y}(0), \bar{Y}(0)) \neq 0$. *Assume that there are no* K_2 *-quotient focal points on (0,b]. Then*

$$\bar{g}_1(\bar{X}(t), \bar{X}(t)) \geq \bar{g}_2(\bar{Y}(t), \bar{Y}(t)).$$

Moreover, if $\bar{g}_1(\bar{X}(t_0), \bar{X}(t_0)) = \bar{g}_2(\bar{Y}(t_0), \bar{Y}(t_0))$ *for some* $t_0 \in (0, b]$, *then* $\bar{g}_1(\bar{X}(t), \bar{X}(t)) = \bar{g}_2(\bar{Y}(t), \bar{Y}(t))$ *for all* $t \in [0, t_0]$.

3. Curvaturelike Tensor Classes and their Associated C^2-Jacobilike Classes along Null Geodesics

Recall the curvaturelike tensor fields which we used to prove the focal Rauch comparison theorem for timelike geodesics in the case that $dim M_1 > dim M_2$ as in [6].

Similarly, we define a continuous *curvaturelike tensor class* $\tilde{R}$ for $TM|_\beta/[\beta']$ by a linear map

$\tilde{R}(., \beta'(t))\beta'(t) : T_{\beta(t)}M/[\beta'(t)] \longrightarrow T_{\beta(t)}M/[\beta'(t)]$ for each $t \in [0, b]$ such that

$\bar{g}(\tilde{R}(\bar{v}, \beta'(t))\beta'(t), \bar{w}) = \bar{g}(\tilde{R}(\bar{w}, \beta'(t))\beta'(t), \bar{v})$

and

$\tilde{R}(\beta'(t),\beta'(t))\beta'(t) = 0.$

Moreover, we define a *C^2-Jacobilike class* of $(M,\bar{g},\tilde{R})$ to be a vector class $\bar{J}$ along β which satisfies

(1) $\bar{J}(0) \in \pi(T_{\beta(0)}K)$,

(2) $\bar{S}_{c'(t)}\bar{J}(0) + \bar{J}'(0) \in \pi((T_{\beta(0)}K)^{\perp})$,

(3) $\bar{J}'' + \tilde{R}(\bar{J},\beta')\beta' = [\beta']$

where $\bar{J}''$ is the covariant differentiation of second order with respect to the given Levi-Civita connection $\bar{\nabla}$ along β and $\tilde{R}$ is the given curvaturelike tensor class for $TM|_{\beta'}$.

As in the timelike case, given any $\bar{v},\bar{w} \in T_{\beta(t_0)}M/[\beta'(t_0)]$, there exists a unique C^2-Jacobilike class $\bar{J}$ along β with $\bar{J}(t_0) = \bar{v}$, $\bar{J}'(t_0) = \bar{w}$.

Moreover, if $\tilde{R}$ is a curvaturelike tensor class along β and $\bar{J}$ is a C^2-Jacobilike class of $(M,\bar{g},\tilde{R})$ along β, then we may show that

(1) $\bar{g}(\bar{J},\beta')|_t$ is an affine function,

(2) if $\bar{J}(t_0) = [\beta'(t_0)]$ and $\bar{J}(t_1) = [\beta'(t_1)], t_0 \neq t_1$, we have $\bar{J} \in \chi(\beta)$.

Thus we may show that the maximality theorem, index comparison theorem, and Rauch comparison theorem for curvaturelike tensor classes and for the associated C^2-Jacobilike classes in the case that $dimM_1 \leq dimM_2$ are still available as given in the timelike case, cf.[7].

Now define a vector bundle $V|_\beta$ along β which is perpendicular to the spacelike submanifold K given by

$V|_{\beta(t)} = T_{\beta(t)}M \oplus \mathbf{R}^l$

with the metric $g_V = g+ <,>$ where $<,>$ is the usual Euclidean metric on $\mathbf{R}^l$ for some $l > 0$. Hence we may define a subbundle $V|_\beta/[\beta']$ of $V|_\beta$ given by

$V|_{\beta(t)}/[\beta'(t)] = T_{\beta(t)}M/[\beta'(t)] \oplus \mathbf{R}^l$

with the metric $g_{V/[\beta']} = \bar{g}+ <,>$. Thus we may define a curvaturelike tensor class $\tilde{R}$ on $V|_\beta/[\beta']$ by

$\tilde{R}(\bar{u} \oplus w,\beta'(t))\beta'(t) = \bar{R}(\bar{u},\beta'(t))\beta'(t) + m(t)w$

for any $\bar{u} \oplus w \in V|_{\beta(t)}/[\beta'(t)]$, where m(t) is the minimum value of $\bar{g}(\bar{R}(\bar{u},\beta'(t))\beta'(t),\bar{u})$for all $\bar{u} \in (\beta'(t))^{\perp}/[\beta'(t)]$ with $\bar{g}(\bar{u},\bar{u}) = 1$ and $\bar{R}$ is the curvature tensor class on $T_{\beta'(t)}/[\beta'(t)]$ with respect to the Levi-Civita connection $\bar{\nabla}$ on $(M,\bar{g})$.

Now we define a subbundle $\bar{K}$ of $V|_{\beta(0)}/[\beta'(0)]$ by $\bar{K} = \pi(T_{\beta(0)}K) \oplus \mathbf{R}^l$.

Then we may define the second fundamental operator $\tilde{S}_{\beta'(0)}$ on $\bar{K}$ by
$\tilde{S}_{\beta'(0)}(\bar{v} \oplus w) = \bar{S}_{\beta'(0)}\bar{v} \oplus \eta w$
for any $\bar{v} \oplus w \in \bar{K}$, where η is the minimum value of $\bar{g}(\bar{S}_{\beta'(0)}\bar{v}, \bar{v})$ for $\bar{v} \in \pi(T_{\beta(0)}K)$ with $\bar{g}(\bar{v}, \bar{v}) = 1$. Thus, for any $\bar{v} \oplus w \in \bar{K}$ with $g_{V/[\beta']}(\bar{v} \oplus w, \bar{v} \oplus w) = 1$,

$$\begin{aligned} \eta &= \eta(\bar{g}(\bar{v}, \bar{v}) + < w, w >) \\ &\leq \bar{g}(\bar{S}_{\beta'(0)}\bar{v}, \bar{v}) + \eta < w, w > \\ &= g_{V/[\beta']}(\tilde{S}_{\beta'(0)}\bar{v} \oplus w, \bar{v} \oplus w). \end{aligned}$$

Moreover, for each $t \in (0, b]$ and for any $\bar{v} \oplus w \in V|_{\beta(t)]}/[\beta'(t)]$ with $g_{V/[\beta']}(\bar{v} \oplus w, \bar{v} \oplus w) = 1$,

$$\begin{aligned} m(t) &= m(t)(\bar{g}(\bar{v}, \bar{v}) + \eta < w, w >) \\ &\leq \bar{g}(\bar{R}(\bar{v}, \beta'(t))\beta'(t), \bar{v}) + m(t) < w, w > \\ &= g_{V/[\beta']}(\tilde{R}(\bar{v} \oplus w, \beta'(t))\beta'(t), \bar{v} \oplus w). \end{aligned}$$

Hence, we may obtain the following results from the conditions ($\bar{\mathbf{R}}$) and ($\bar{\mathbf{S}}$).

LEMMA 3.1. *η is less than or equal to the minimum value of $g_{V/[\beta']}$ $(\tilde{S}_{\beta'(0)}\bar{v} \oplus w, \bar{v} \oplus w)$ for $\bar{v} \oplus w \in \bar{K}$ with $g_{V/[\beta']}(\bar{v} \oplus w, \bar{v} \oplus w) = 1$. For each $t \in (0, b]$, $m(t)$ is less than or equal to the minimum value of $g_{V/[\beta']}(\tilde{R}(\bar{v} \oplus w, \beta'(t))\beta'(t), \bar{v} \oplus w)$ for $\bar{v} \oplus w \in V|_\beta/[\beta'(t)]$ with $g_{V/[\beta']}(\bar{v} \oplus w, \bar{v} \oplus w) = 1$.*

Given a vector class $\bar{J}_1 \in TM|_\beta/[\beta']$ and a vector field $J_2 \in \mathbf{R}^l|_\beta$, we may lift them to $TM|_\beta/[\beta'] \oplus \mathbf{R}^l|_\beta$ and obtain the vector class $\bar{J} = (\bar{J}_1, 0) + (0, J_2)$ in $TM|_\beta/[\beta'] \oplus \mathbf{R}^l|_\beta$. Then the connection on $TM|_\beta/[\beta'] \oplus \mathbf{R}^l|_\beta$ associated to the metric $g_{V/[\beta']}$ satisfies $\bar{J}' = (\bar{J}_1', 0) + (0, J_2')$ for such a sum of lifted vector classes.

Moreover, we see that

$$\begin{aligned} &\bar{J}'(0) + \tilde{S}_{\beta'(0)}\bar{J}(0) \\ =&(\bar{J}_1'(0) + \bar{S}_{\beta'(0)}\bar{J}_2(0)) \oplus (J_2'(0) + \eta J_2(0)) \\ \in&\pi((T_{\beta(0)}K)^\perp) \oplus \{0\}. \end{aligned}$$

Hence we have $\bar{J}_1'(0)+\bar{S}_{\beta'(0)}\bar{J}_1(0) \in \pi((T_{\beta(0)}K)^{\perp})$ and $J_2'(0)+\eta J_2(0)$ $=0$.

Thus a $C^2\bar{K}$-Jacobilike class in $TM|_\beta/[\beta'] \oplus \mathbf{R}^l|_\beta$ may be split nicely into a C^2 K-Jacobi class in $TM|_\beta/[\beta']$ and a Jacobi field in $\mathbf{R}^l|_\beta$ respectively as follows.

LEMMA 3.2. *Let* $\pi_1 : V|_\beta/[\beta'] \longrightarrow TM|_\beta/[\beta']$ *and* $\pi_2 : V|_\beta/[\beta'] \longrightarrow \mathbf{R}^l|_\beta$ *denote the projections. Then* $\bar{J}$ *is a* $C^2\bar{K}$*- Jacobilike class for* $(V|_\beta/[\beta'], g_{V/[\beta']}, \tilde{R})$ *iff* $\bar{J}_1 = \pi_1(\bar{J})$ *is a* C^2 *K-Jacobi class for* $(M, \bar{g}, \bar{R})$ *and* $J_2 = \pi_2(\bar{J})$ *is a* C^2*-Jacobi field for* $(\mathbf{R}^l|_\beta, <,>, m(t)I)$.

4. Extension of the Null Rauch Comparison Theorem

Let (M_i, g_i) be arbitrary Lorentzian manifolds and let $\beta_i : [0,b] \longrightarrow (M_i, g_i)$ be null geodesics perpendicular at $\beta_i(0)$ to the spacelike submanifold K_i for i = 1,2. Then we first need the following comparison of the relative location of the quotient K_1- and K_2-focal points on $\beta_1(t)$ and $\beta_2(t)$ for $t \in (0,b]$ before proving the focal Rauch comparison theorem for null geodesics in the case that $dimM_1 > dimM_2$. Moreover, we require $k_1 > 0$ and an additional hypothesis comparing the second fundamental operators $\bar{S}_{\beta_1'(0)}$ and $\bar{S}_{\beta_2'(0)}$ of K_1 and K_2 respectively.

THEOREM 4.1. *Assume that if* $dimK_1 = k_1 > 0$*, then* $dimK_2 = k_2 > 0$*. Suppose the conditions* $(\tilde{\mathbf{R}})$ *and* $(\bar{\mathbf{S}})$ *hold. Then if there are no* K_2*-quotient focal points on (0,b], there are no* K_1*-quotient focal points on (0,b].*

Proof. Suppose there is a K_1-quotient focal point at $t_0 \in (0,b]$. We will show that there is a K_2-quotient focal point on (0,b], in contradiction.

Assume first that $dimM_1 = n_1 \leq dimM_2 = n_2$. Let $\bar{Y}$ be a nontrivial K_1-Jacobi class in $\chi(\beta_1, K_1)$ such that $\bar{Y}(t_0) = [\beta_1'(t_0)]$. Choose a K_1-Jacobi field $Y_1 \in V^{\perp}(\beta_1, K_1)$ with $\pi(Y_1) = \bar{Y}$. Then $Y_1(t_0) = \lambda\beta_1'(t_0)$ for some $\lambda \in \mathbf{R}$. Then $Y(t) = Y_1(t) - (\lambda t/t_0)\beta_1'(t)$ is a nontrivial K_1-Jacobi field in $V^{\perp}(\beta_1', K_1)$ with $Y(t_0) = 0$ and $\pi(Y) = \bar{Y}$.

Extend Y to be 0 from t_0 to b. Then $Y \in V_0^{\perp}(\beta_1, K_1)$ and thus $I_{(b,K_1)}(Y,Y) = 0$.

Let $e_1 = Y(0)/g_1(Y(0),Y(0))^{1/2}$ if $Y(0) \neq 0$ and let e_1 be an arbitrary unit vector in $T_{\beta_1(0)}K_1$ if $Y(0) = 0$. Then we may find an pseudo-orthonormal parallel spacelike vector fields $\{E_i\}_{i=1,2,\ldots,n_1-2}$ in $V^\perp(\beta_1)$ such that $E_1(0) = e_1$. Hence $\bar{Y}(t) = \sum_{i=1}^{n_1-2} h_i(t)\bar{E}_i(t)$. where $\pi(E_i) = \bar{E}_i$ for piecewise smooth functions h_i on [0,b] with $h_i(0) = 0$ for $2 \leq i \leq n_1 - 2$.

Let $\{\tilde{E}_i\}_{i=1,2,\ldots,n_2-2}$ be pseudo-orthonormal parallel vector fields in $V^\perp(\beta_2)$ with

$$\tilde{E}_1(0) = j(e_1) \in T_{\beta_2(0)}K_2$$

where $j : T_{\beta_1(0)}M_1 \to T_{\beta_2(0)}M_2$ is an injective linear isometry such that $j(\beta_1'(0)) = \beta_2'(0)$ and $j(e_1) \in T_{\beta_2(0)}K_2$.

Now we may define

$$\bar{Z}(t) = \sum_{i=1}^{n_1-2} h_i(t)\pi(\tilde{E}_i(t)).$$

Since $Y(0) = g_1(Y(0),Y(0))^{1/2}e_1$, we obtain $\sum_{i=1}^{n_1-2} h_i(0)\bar{E}_i(0) = g_1(Y(0),Y(0))^{1/2}\pi(e_1)$. Hence, $h_1(0) = g_1(Y(0),Y(0))^{1/2}$, $h_2(0) = h_3(0) = \ldots = h_{n_1-2}(0) = 0$.

Moreover, we may check that $\bar{Z}(0) \in \pi(T_{\beta_2(0)}K_2)$ and that $\bar{Z}(b) = 0$. Thus, $\bar{Z} \in \chi_0(\beta_1, K_1)$.

Since $\bar{g}_1(\bar{Y}(t),\bar{Y}(t)) = \bar{g}_2(\bar{Z}(t),\bar{Z}(t))$ for all t, our hypotheses imply

$$\bar{g}_1(\bar{R}_1(\bar{Y},\beta_1')\beta_1',\bar{Y})|_t \leq \bar{g}_2(\bar{R}_2(\bar{Z},\beta_2')\beta_2',\bar{Z})|_t$$

and

$$\bar{g}_1(\bar{S}_{\beta_1'(0)}\bar{Y}(0),\bar{Y}(0)) \leq \bar{g}_2(\bar{S}_{\beta_2'(0)}\bar{Z}(0),\bar{Z}(0)).$$

Hence,

$0 = \bar{I}_{(b,K)}(\bar{Y},\bar{Y}) \leq \bar{I}_{(0,b)}(\bar{Z},\bar{Z})$.

This contradicts Corollary 2.2.

Note that this same argument is valid for curvaturelike tensor classes along β_1 and β_2.

Suppose now that $dimM_1 > dimM_2$ and let $l = dimM_1 - dimM_2$. Recall that a metric $g_{V/[\beta_2']}$ and the curvaturelike tensor field $\tilde{R}$ on the bundle $V/[\beta_2']$ along β_2, and the second fundamental operator $\tilde{S}_{\beta_2'(0)}$ on $\tilde{K} = \pi(T_{\beta_2(0)}K_2) \oplus \mathbf{R}^l$ as in §3 may be given by

$$g|_{V/[\beta_2']} = \bar{g} + <,>,$$

$$\tilde{R}(\bar{v} \oplus w, \beta_2'(t))\beta_2'(t) = \bar{R}_2(\bar{v}, \beta_2'(t))\beta_2'(t) + m(t)w,$$

and

$$\tilde{S}_{\beta_2'(0)}(\bar{v} \oplus w) = \bar{S}_{\beta_2'(0)}\bar{v} \oplus \eta w.$$

If $dimK_1 > 0$, we have assumed $dimK_2 > 0$, and hence $dim\tilde{K} = dimK_2 + l > 0$. Thus we need to check that $(M_1, \bar{g}_1, \bar{R}_1)$ and $(V/[\beta_2'], g_{V/[\beta']}, \tilde{R})$ satisfy the curvature and the second fundamental operator hypotheses of this theorem so that the first part of the proof($dimM_1 \leq dimM_2$) may be applied to these spaces.

By hypotheses of $(\bar{\mathbf{R}})$ and $(\bar{\mathbf{S}})$ and using Lemma 3.1, the maximum value of $\bar{g}_1(\bar{S}_{\beta_1'(0)}\bar{u}, \bar{u})$ for $\bar{u} \in \pi(T_{\beta_1(0)}K_1)$ with $\bar{g}_1(\bar{u}, \bar{u}) = 1$ is less than or equal to the minmum value of $g_{V/[\beta_2']}(\tilde{S}_{\beta_2'(0)}\bar{v} \oplus w, \bar{v} \oplus w)$ for $\bar{v} \oplus w \in \pi(T_{\beta_2(0)}K_2) \oplus \mathbf{R}^l = \tilde{K}$ with $\bar{g}_2(\bar{v}, \bar{v}) = g_{V/[\beta_2']}(\bar{v} \oplus w, \bar{v} \oplus w) = 1$. Moreover, for each $t \in [0, b]$, the maximum value of $\bar{g}_1(\bar{R}_1(\bar{u}, \beta_1'(t))\beta_1'(t), \bar{u})$ for $\bar{u} \in (\beta_1'(t))^{\perp}/[\beta_1'(t)]$ with $\bar{g}_1(\bar{u}, \bar{u}) = 1$ is less than or equal to the minimum value of $g_{V/[\beta']}(\tilde{R}(\xi, \beta_2'(t))\beta_2'(t), \xi)$ for $\xi \in V|_{\beta_2'(t)}/[\beta_2'(t)]$ with $g_{V/[\beta_2']}(\xi, \xi) = 1$.

Hence, it is enough to show that if $(M_2, \bar{g}_2)$ has no quotient K_2-focal points, then $(V/[\beta_2'], g_{V/[\beta_2']})$ has no quotient $\tilde{K}$-focal points since the first part of the proof (even for curvaturelike tensor classes) may be applied to $(M_1, \bar{g}_1, \bar{R}_1)$ and $(V/[\beta_2'], g_{V/[\beta_2']}, \tilde{R})$ as $dimM_1 = dimV$.

Suppose that $\bar{J}$ is a nontrivial C^2 $\tilde{K}$-Jacobilike class in $(V/[\beta_2'], g_{V/[\beta_2']}, \tilde{R})$ with $\bar{J}(t_0) = ([\beta_2'(t_0)], 0)$ for some $t_0 \in (0, b]$. By Lemma 3.2, we decompose $\bar{J} = (\bar{J}_1, J_2)$ where $\bar{J}_1$ is a C^2 K_2-Jacobi class in $(M_2, \bar{g}_2, \bar{R}_2)$ and J_2 is a Jacobi field in $(\mathbf{R}^l, <,>, m(t)I)$.

Since $\bar{J}(t_0) = ([\beta_2'(t_0)], 0)$ implies that $\bar{J}_1(t_0) = [\beta_2'(t_0)]$ and $J_2(t_0) = 0$, and since M_2 has no quotient K_2-focal points, we obtain $\bar{J}_1 = [\beta_2']$. It means that $J_2 \neq 0$ on the Euclidean factor. Moreover, we have a conjugate point equation

$$J_2'' + m(t)J_2 = 0$$

with $J_2(0) = J_2(t_0) = 0$ and $J_2'(0) + \eta J_2(0) = 0$. Since $J_2 \neq 0$, we have $J_2'(t_0) \neq 0$.

Now let $E_1, E_2, ..., E_l$ be parallel fields along β_2 in the bundle $(\mathbf{R}^l|_{\beta_2'}, <,>)$. Writing

$$J_2 = \sum_{j=1}^{l} \psi_j(t) E_j,$$

we obtain k-scalar equations for $t \in [0, b], 1 \leq j \leq l$,

$$\psi_j''(t) + m(t)\psi_j(t) = 0$$
$$\psi_j(0) = \psi_j(t_0) = 0, \psi_j'(0) + \eta\psi_j(0) = 0.$$

Since $J_2'(t_0) \neq 0$, we also have $\psi_i'(0) \neq 0$ for some i. For this particular i, extend ψ_i to a piecewise smooth function $\psi : [0, b] \longrightarrow \mathbf{R}$ with $\psi(t) = \psi_i(t)$ for $0 \leq t \leq t_0$ and 0 for $t \geq t_0$.

Then the index form on $(\mathbf{R}, <,>, m(t), \{0\})$ may be given as follows; for the field $Z = \psi(t)d/dt$,

$$\begin{aligned} I_{(b,\mathbf{R})}(Z, Z) &= < \eta Z(0) + Z'(0), Z(0) > - < Z'(t_0), Z(t_0) > \\ &\quad - \int_0^{t_0} < Z'' + m(t)Z, Z > dt \\ &= 0. \end{aligned}$$

Select any $e_1 \in T_{\beta_2(0)}K_2$ and parallel translate along β_2 to a parallel field E_1. If we take a linear isometry $j : T_0\mathbf{R} \longrightarrow T_{\beta_2(0)}K_2$ with $j(d/dt|_0) = e_1$, then the associated linear isometry given by parallel translation into $V^{\perp}(\beta_2)$ satiesfies

$$Y = \phi(Z) = \psi(t)E_1 \in V_0^{\perp}(\beta_2, K_2).$$

Also, Y is nontrivial since $\psi'(0) \neq 0$. Hence we have a nontrivial $\bar{Y}$ with $\pi(Y) = (\bar{Y}) \in \chi_0(\beta_2, K_2)$.

Thus, we have as before, using $Y(0) = 0$ after the inequality,

$$\begin{aligned} I_{(b,\mathbf{R})}(Z, Z) &\leq g_2(S_{\beta_2'(0)}Y(0), Y(0)) \\ &\quad - \int_0^b [g_2(Y', Y') - g_2(R_2(Y, \beta_2')\beta_2', Y)]dt \\ &= I_{(b,K_2)}(Y, Y) \end{aligned}$$

Thus,

$$0 = I(Z, Z) \leq I_{(b,K)}(Y, Y) = \bar{I}_{(b,K)}(\bar{Y}, \bar{Y})$$

Since $\bar{Y} \neq [\beta']$ and since (M_2, g_2) has no quotient K_2- focal points, we have $\bar{I}_{(b,K_2)}(\bar{Y}, \bar{Y}) < 0$, in contradiction to Corollary 2.2. Therefore $\psi(t) = 0$ in $(\mathbf{R}, <,>, m(t), \{0\})$ and we conclude that $V/[\beta_2']$ has no quotient K_2-focal points as required.

Finally, we show that the quotient focal Rauch comparison theorem for null geodesics is still valid for $dim M_1 > dim M_2$ as in the timelike case, cf.[7].

THEOREM 4.2. *Let (M_i, g_i) be arbitrary Lorentzian manifolds of $dim M_i = n_i \geq 3$ with $n_1 > n_2$ and let $\beta_i : [0, b] \to M_i$ be null geodesics perpendicular to the spacelike submanifolds K_i, i = 1,2. For each $t \in [0, b]$ suppose the conditions $(\bar{\mathbf{R}})$ and $(\bar{\mathbf{S}})$ hold and suppose $dim K_1 = k_1 > 0$, $dim K_2 = k_2 = n_2 - 2$, and let $\bar{X} \in \chi(\beta_1, K_1)$, $\bar{Y} \in \chi(\beta_2, K_2)$ be quotient K_1- and K_2-Jacobi classes respectively such that*

$$\bar{g}_1(\bar{X}(0), \bar{X}(0)) = \bar{g}_2(\bar{Y}(0), \bar{Y}(0)) \neq 0.$$

Assume that there are no quotient K_2-focal points on (0,b]. Then

$$\bar{g}_1(X(t), \bar{X}(t)) \geq \bar{g}_2(\bar{Y}(t), \bar{Y}(t)).$$

Moreover, if $\bar{g}_1(\bar{X}(t_0), \bar{X}(t_0)) = \bar{g}_2(\bar{Y}(t_0), \bar{Y}(t_0))$ for some $t_0 \in (0, b]$, then $\bar{g}_1(\bar{X}(t), \bar{X}(t)) = \bar{g}_2(\bar{Y}(t), \bar{Y}(t))$ for all $t \in [0, t_0]$.

Proof. We proved this theorem for $n_1 \leq n_2$ in Theorem 2.4. The proof is, moreover, valid for curvaturelike tensor classes along β_1 and β_2 as in the timelike case, cf.[7].

Now assume that $n_1 > n_2$ and construct a triple $(V/[\beta_2'], g_{V/[\beta_2']}, \tilde{K})$ as in the proof of Theorem 4.1. By the condition $(\bar{\mathbf{R}})$, the maximum value of $\bar{g}_1(\bar{R}_1(\bar{u}, \beta_1'(t))\beta_1'(t), \bar{u})$ for $\bar{u} \in (\beta_1'(t))^\perp/[\beta_1'(t)]$ with $\bar{g}_1(\bar{u}, \bar{u}) = 1$ is less than or equal to the minimum value of $g_{V/[\beta_2']}(\tilde{R}(\xi, \beta_2'(t))\beta_2'(t),$
$\xi)$ for $\xi \in V|_{\beta_2(t)}/[\beta_2'(t)]$ with $g_{V/[\beta_2']}(\xi, \xi) = 1$, for each $t \in [0, b]$.

Also, by the condition ($\bar{\mathbf{S}}$), the maximum value of $\bar{g}_1(\tilde{S}_{\beta_1'(0)}\bar{u},\bar{u})$ for $\bar{u} \in \pi(T_{\beta_1(0)}K_1)$ with $\bar{g}_1(\bar{u},\bar{u}) = 1$ is less than or equal to the minimum value of $g_{V/[\beta_2']}(\bar{v}\oplus w,\bar{v}\oplus w)$ for $\bar{v}\oplus w \in \bar{K}$ with $g_{V/[\beta_2']}(\bar{v}\oplus w,\bar{v}\oplus w) = 1$.

Since $dim K_1 = k_1 > 0$ and $dim\bar{K} = dim\bar{K}_2 + (n_1 - n_2) = n_1 - 2$ and since $dim M_1 = n_1$, $dim V/[\beta_2'] = n_2 + (n_1 - n_2) = n_1$, the theorem holds for (M_1, g_1, K_1) and $(V/[\beta_2], g_{V/[\beta_2']}, \bar{K})$ by Theorem 2.4. Thus we may obtain this theorem for (M_1, g_1, K_1) and (M_2, g_2, K_2) as follows;

Let $\bar{X} \in \chi(\beta_1, K_1)$ and $\bar{Y} \in \chi(\beta_2, K_2)$ be quotient K_1- and K_2-Jacobi classes with

$$\bar{g}_1(\bar{X}(0),\bar{X}(0)) = \bar{g}_2(\bar{Y}(0),\bar{Y}(0)) \neq 0.$$

Since,

$\tilde{S}_{\beta_2'(0)}(\bar{Y}(0),0)+(\bar{Y}'(0),0) = (\tilde{S}_{\beta_2'(0)}\bar{Y}(0)+\bar{Y}'(0),0) \in \pi((T_{\beta_2(0)}K_2)^{\perp})$ $\oplus\{0\}$, $(\bar{Y},0)$ is the quotient $\bar{K}$-Jacobi class for $V/[\beta_2']$ such that

$\bar{g}_1(\bar{X}(0),\bar{X}(0)) = \bar{g}_2(\bar{Y}(0),\bar{Y}(0)) = g_{V/[\beta_2']}((\bar{Y}(0),0),(\bar{Y}(0),0))$.

Since $(V/[\beta_2'], g_{V/[\beta_2']}, \bar{K})$ has no quotient $\bar{K}$- focal points on β_2, by Theorem 2.4, applied to $(M_1, \bar{g}_1, K_1)$ and $(V/[\beta_2'], g_{V/[\beta_2]}, \bar{K})$, we obtain

$$\begin{aligned}\bar{g}_1(\bar{X}(t),\bar{X}(t)) &\geq g_{V/[\beta_2']}((\bar{Y}(t),0),(\bar{Y}(t),0))\\ &= \bar{g}_2(\bar{Y}(t),\bar{Y}(t)).\end{aligned}$$

Moreover, if $\bar{g}_1(\bar{X}(t_0),\bar{X}(t_0)) = \bar{g}_2(\bar{Y}(t_0),\bar{Y}(t_0))$ for some $t_0 \in (0,b]$. Then

$$\bar{g}_1(\bar{X}(t_0),\bar{X}(t_0)) = g_{V/[\beta_2]}((\bar{Y}(t_0),0),(\bar{Y}(t_0),0)).$$

Thus, by Theorem 2.4, for all $t \in [0,t_0]$,

$$\bar{g}_1(\bar{X}(t),\bar{X}(t)) = g_{V/[\beta_2']}((\bar{Y}(t),0),(\bar{Y}(t),0)).$$

Therefore, for all $t \in [0,t_0]$,

$$\bar{g}_1(\bar{X}(t),\bar{X}(t)) = \bar{g}_2(\bar{Y}(t),\bar{Y}(t)).$$

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