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ON A STRUCTURE ϕ SATISFYING $(\phi^2+1)(\phi^2+a)=0$

JONG-IK HYUN

In [4],Yano,Houh and Chen have consider the structure defined by a tensor field ϕ of type (1,1) satisfying $\phi^4+\phi^2=0$ or $\phi^4-\phi^2=0$, respectively, and they studied the existence of those structures. In [2],Nikić is introduces the $\phi(-1,-a)$-structure satisfying $(\phi^2-1)(\phi^2-a)=0$. In this paper,we want to consider the structure defined by a tensor field ϕ of type (1,1) satisfying $(\phi^2+1)(\phi^2+a)=0$. Moreover,we consider the generalization of the above structure which satisfies the equation;

$$(\phi^2+a_1)(\phi^2+a_2)\cdots(\phi^2+a_k)=0,$$

where $a_1,a_2,\cdots,a_k$ are distinct real numbers.

I. Perliminaries

Let M be an n-dimensional differentiable manifold with (f,U,V,u,v,λ) -structure. Then there exist a tensor field f of type (1,1) on M, vector fields U and V, 1-forms u and v, and a function λ such that

$$\begin{aligned}
&f^2=-1+u\otimes U+v\otimes V,\\
&fU=-\lambda V,\quad fV=\lambda U,\\
&u\circ f=\lambda v,\quad v\circ f=-\lambda u,\\
&u(U)=v(V)=1-\lambda^2,\\
&u(V)=v(U)=0.
\end{aligned}\tag{1.1}$$

Applying f to the first equation of (1.1),we get

$$f^3+f=\lambda(-u\otimes V+v\otimes U).\tag{1.2}$$

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If $\lambda \neq 0$, then the (f, U, V, u, v, λ)-structure is not an f-structure defined by Yano [3]. Again,applying f to (1.2) and making use of (1.1), we get

$$(f^2+1)\left(f^2+\lambda^2\right)=0. \tag{1.3}$$

Then the (f, U, V, u, v, λ)-structure is not a $\phi(-1,-a)$-structure defined by Nikić [2].

II. Matric $\phi(1, a)$-structure

Let M^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field ϕ of type (1.1) and of class c^∞ such that

$$(\phi^2+1)(\phi^2+a)=0, \quad a \in R^+, a \neq 1. \tag{2.1}$$

For a differentiable manifold with a structure which satisfies the equation (2.1), we say that it admits a $\phi(1, a)$-structure. Let

$$l=\frac{(\phi^2+a)}{(a-1)}, \quad m=\frac{(\phi^2+1)}{(1-a)}, \tag{2.2}$$

then we have

$$l+m=1, \quad l^2=l, \quad m^2=m, \quad lm=ml=0.$$

Thus the operators l and m applied to the tangent space at a point of M^n are complementary projection operators. Let L and M be the complementary distributions corresponding to the operators l and m respectively.

From (2.2) we get

$$\begin{aligned} &\phi^2 l=-l, \quad \phi^2 m=-am \quad \text{and} \\ &f=l-m=\frac{(2\phi^2+a+1)}{(a-1)}. \end{aligned} \tag{2.3}$$

Then it clear that $f \neq 1, f^2=1$ and the manifold M^n admits an almost product structure f.

THEOREM 2.1. *A differentiable manifold M^n with $\phi(1,a)$-structure is of even dimensional.*

Proof. Let p be a point of M^n. Then we have from (2.3)

$$\phi^2 X = -X, \quad X \in L_p,$$

which shows that ϕ is an almost complex structure in the subspace L_p of $T_p(M^n)$ at p and L_p is of even dimensional,where we put $\dim L_p = 2k$.

From the second equation of (2.3),we get

$$\phi^2 X = -aX, \quad X \in M_p,$$

which shows that $\phi' = \dfrac{\phi}{\sqrt{a}}$ is an almost complex structure in the subspace M_p of $T_p(M^n)$ at p and M_p is of even dimensional. Thus $T_p(M^n)$ is of even dimensional,where we put $n = 2m$.

We now introduce a local coordinate system in the manifold M^n and denote by ϕ_j^i, l_j^i, m_j^i the local components of the tensors ϕ, l, m, respectively. We also introduce a positive definite Riemannian metric g in M^n and take $2k$ mutually orthogonal unit vectors $u_a^h (a, b, c, \cdots = 1, 2, \cdots, 2k)$ in L and $2(m-k)$ mutually orthogonal unit vectors u_A^h $(A, B, C, \cdots = 2k+1, \cdots, 2m)$ in M.

Then we have

$$(2.4) \qquad l_i^h u_a^i = u_a^h, \quad l_i^h u_A^i = 0, \quad m_i^h u_a^i = 0, \quad m_i^h u_A^i = u_A^h.$$

If we denote by (v_i^a, v_i^A) the matrix inverse to (u_b^j, u_B^j), then both v_i^a and v_i^A are components of linearly independent covarient vectors and satisfy the relations

$$(2.5) \qquad v_i^a u_b^i = \delta_b^a, \quad v_i^a u_B^i = 0, \quad v_i^A u_b^i = 0, \quad v_i^A u_B^i = \delta_B^A,$$

$$(2.6) \qquad v_i^a u_a^h + v_i^A u_A^h = \delta_i^h.$$

If we put

$$(2.7) \qquad h_{ji} = v_j^a v_i^a + v_j^A v_i^A,$$

then h_{ji} is a globally well-defined positive definite Reimannian metric with respect to which (v_b^h, v_B^h) form an orthogonal frame such that $v_j^a = h_{ji}u_a^i, v_j^A = h_{ji}u_A^i$.

From (2.4) and (2.5),we get

$$l_j^h v_h^a = v_j^a, \quad l_j^h v_h^A = 0, \quad m_j^h v_h^a = 0, \quad m_j^h v_h^A = v_j^A. \tag{2.8}$$

On the other hand, from the first equation of (2.4), we get

$$l_j^h v_i^a u_a^j = v_i^a u_a^h, \quad l_j^h \left(\delta_i^j - v_i^A u_A^j\right) = v_i^a u_a^h,$$

from which

$$l_i^h = v_i^a u_a^h. \tag{2.9}$$

Similarly we get

$$m_i^h = v_i^A u_A^h. \tag{2.10}$$

If we put

$$l_{ji} = l_j^t h_{ti}, \quad m_{ji} = m_j^t h_{ti} \tag{2.11}$$

We find from (2.7),(2.9) and (2.10)

$$l_{ji} = v_j^a v_i^a, \quad m_{ji} = v_j^A v_i^A \tag{2.12}$$

$$l_{ji} = l_{ij}, \quad m_{ji} = m_{ij}, \quad l_{ij} + m_{ji} = h_{ji}. \tag{2.13}$$

We can easily verify the following relations:

$$\begin{aligned} &l_j^t l_i^s h_{ts} = l_{ji}, \quad l_j^t m_i^s h_{ts} = 0, \\ &m_j^t m_i^s h_{ts} = m_{ji}. \end{aligned} \tag{2.14}$$

For any vectors X and Y with components X^i, Y^i let us put

$$\begin{aligned} &m^*(X,Y) = m_{ts}X^tY^s, \quad h(X,Y) = h_{ts}X^tY^s, \\ &g(X,Y) = \frac{1}{2}[h(X,Y) + h(\phi X, \phi Y) + (a-1)m^*(X,Y)]. \end{aligned} \tag{2.15}$$

Then we have

$$h(u_a,u_A)=m^*(u_a,u_A)=0,\quad g(u_a,u_A)=0.$$

Then the distributions L and M are orthogonal with respect to the metric g. It is easy to verify by using (2.10) and (2.13) that

$$\begin{aligned} m^*(u_a,u_b)&=m^*(\phi u_a,\phi u_b)=0,\\ h(\phi^2u_a,\phi^2u_b)&=h(u_a,u_b). \end{aligned}$$

These equations lead to the following:

$$g(\phi X,\phi Y)=g(X,Y)\quad X,Y\in L. \tag{2.16}$$

Next,for any vector fields u_A and u_B in M, we have from (2.10) and (2.13)

$$\begin{aligned} h(u_A,u_B)&=m^*(u_A,u_B)\\ h(\phi u_A,\phi u_B)&=m^*(\phi u_A,\phi u_B),\\ h(\phi^2u_A,\phi^2u_B)&=h(-au_A,-au_B)=a^2m^*(u_A,u_B). \end{aligned}$$

These equations lead to the following:

$$g(\phi X,\phi Y)=ag(X,Y),\quad X,Y\in M. \tag{2.17}$$

Thus we have

THEOREM 2.2. *Let M^n be an n-dimensional manifold with $\phi(1,a)$-structure. Then there exists a positive definite Riemannian metric g with respect to which L and M are mutually orthogonal and such that (2.16) and (2.17) hold.*

For any vector fields X and Y in M^n, from (2.16) and (2.17) we get

$$\begin{aligned} g(\phi X,\phi Y)&=g(\phi lX+\phi mX,\phi lY+\phi mY)\\ &=g(lX,lY)+ag(mX,mY), \end{aligned}$$

from which

$$g(\phi X,\phi Y)=g(X,Y)+(a-1)g(mX,mY), \tag{2.18}$$

Thus we have

THEOREM 2.3. *Let M^n be an n-dimensional manifold with $\phi(1,a)$-structure. Then there exists a positive definite Riemannian metric g such that*

$$g(\phi X, \phi Y) = g(X,Y) + (a-1)g(mX, mY)$$

for any vector fields X and Y on M^n.

Next,from (2.18) we get

$$\begin{aligned} g(\phi X, Y) &= g(\phi^2 X, \phi Y) + (1-a)(m\phi X, mY) \\ &= g(-X + (1-a)mX, \phi Y) + (1-a)g(\phi X, mY) \\ &= -g(X, \phi Y) + (1-a)g(mX, \phi Y) + (1-a)g(\phi X, mY). \end{aligned}$$

Replacing Y by mY in the last equation,we get

$$g(\phi X, mY) + g(mX, \phi Y) = 0$$

by virtue of $a \in R^+$. Then we have

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.19}$$

Let ω be a tensor field of type (0,2) of M^n defined by

$$\omega(X,Y) = g(\phi X, Y)$$

for any vector fields X and Y of M^n,the we have

$$\omega(X,Y) = -\omega(Y,X), \tag{2.20}$$

that is, ω is a 2-form.

III. Structure group of $\phi(1,a)$-structure

Take a vector e in the distribution L,then the vector ϕe is also in L and perpendicular to e,and moreover ϕe has the same length as e with respect to g. Consequently we can choose $2k$ orthonormal basis in L such that

$$e_1, \cdots, e_k, \quad \phi e_1 = \phi e_{k+1}, \cdots, \phi e_k = e_{2k}.$$

Take a vector e in the distribution M, then the vector ϕe is in M and perpendicular to e. If we put $\phi = \sqrt{a}\phi'$,then ϕ' is an almost complex structure on M, and ϕe has the same length as $\sqrt{a}e$ with respect to g. Consequently we can choose $2(m-k)$ orthonormal basis $\{e_{2k+1},\cdots,e_{2m}\}$ in M such that

$$\phi e_{2k+1} = \sqrt{a}e_{m+k+1},\cdots,\phi e_{m+k} = \sqrt{a}e_{2m}.$$

Then with respect to the orthonormal basis $\{e_1,\cdots,e_{2m}\}$, the tensors g and ϕ have the components:

$$g = \begin{pmatrix} E_k & 0 & 0 & 0 \\ 0 & E_k & 0 & 0 \\ 0 & 0 & E_{m-k} & 0 \\ 0 & 0 & 0 & E_{m-k} \end{pmatrix}$$

$$(3.1) \qquad \phi = \begin{pmatrix} 0 & E_k & 0 & 0 \\ -E_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{a}E_{m-k} \\ 0 & 0 & -\sqrt{a}E_{m-k} & 0 \end{pmatrix}$$

We call such a frame an adapted frame of the $\phi(1,a)$-structure. Now take another adapted frame $\{\bar{e}_1,\cdots,\bar{e}_{2m}\}$ with respect to which the metric g and the structure tensor ϕ have the same components as (3.1). If we put

$$\bar{e}_i = T_i^j e_j \quad (1 \le i,j \le n)$$

then we can easily find that the orthogonal matrix T has the form

$$(3.2) \qquad T = \begin{pmatrix} A_k & B_k & 0 & 0 \\ -B_k & A_k & 0 & 0 \\ 0 & 0 & C_{m-k} & D_{m-k} \\ 0 & 0 & -D_{m-k} & C_{m-k} \end{pmatrix}$$

Thus the structure group of tangent bundle of the manifold M^n can be reduced to $U(k)\times U(m-k)$. Conversely, if the group of tangent bundle of the manifold M^n can be reduced to $U(k)\times U(m-k)$, then we can define a positive definite Riemannian metric g and the tensor ϕ of type

(1.1) as tensor having (3.1) as components with respect to the adapted frames. Then we have

$$(3.3)\qquad \phi^2=\begin{pmatrix} -E_k & 0 & 0 & 0 \\ 0 & -E_k & 0 & 0 \\ 0 & 0 & -aE_{m-k} & 0 \\ 0 & 0 & 0 & -aE_{m-k} \end{pmatrix}$$

and it is easily verified that $(\phi^2+1)(\phi^2+a)=0$. From this we have

THEOREM 3.1. *A necessary and sufficient condition for an n-dimensional manifold admit a tensor field ϕ of type (1,1) defining a $\phi(1,a)$-structure is that the group of tangent bundle of the reduced to the group $U(k)\times U(m-k)$.*

IV. Generalization

In this section,we study to generalize the $\phi(1,a)$-structure. Let M^n be a differentiable manifold of class C^∞ and let there be given a tensor field ϕ of type (1,1) such that

$$(4.1)\qquad (\phi^2+a_1)(\phi^2+a_2)\cdots(\phi^2+a_k)=0$$

where $a_1,a_2,\cdots,a_k$ are distinct positive real numbers. For a manifold M^n with a structure which satisfies such condition we say that it admits a $\phi(a_1,a_2,\cdots,a_k)$-structure. Let

$$(4.2)\qquad l_i=t_i(\phi^2+a_1)\cdots(\phi^2+a_{i-1})(\phi^2+a_{i+1})\cdots(\phi^2+a_k),\quad (i=1,2,\cdots,k)$$

where

$$(4.3)\qquad t_i=\frac{1}{(a_1-a_i)\cdots(a_{i-1}-a_i)(a_{i+1}-a_i)\cdots(a_k-a_i)}$$

Then we have

$$(4.4)\qquad \begin{aligned} &l_i^2=l_i,\quad l_il_j=l_jl_i=0\quad (i\neq j)\\ &l_i+l_2+\cdots+l_k=1\end{aligned}$$

From (4.4) we see that the operators $l_1,l_2\cdots,l_k$ acting to the tangent space at each point of M^n are complementary projection operators. Then there exist the complementary distributions $L_1,L_2,\cdots,L_k$ corresponding to the operators $l_1,l_2,\cdots,l_k$, respectively.

THEOREM 4.1. *A differentiable manifold M^n with $\phi(a_1, a_2, \cdots, a_k)$-structure is of even dimensional.*

Proof. Let p be a point of M^n. From (4.1) and (4.2) we get

$$\phi^2 l_i = -a_i l_i \quad (i = 1, 2, \cdots, k) \tag{4.5}$$

If we put $\phi = \sqrt{a_i}\phi_i$ in the distribution L_i,then we get ${\phi_i}^2 = -1$ and there exists an almost complex structure ϕ_i in $(L_i)_p$. Thus $(L_i)_p$ is of even dimensional and denote by $\dim L_i = 2r_i$. Hence we have

$$n = 2r_1 + 2r_2 + \cdots + 2r_k = 2m.$$

Next,we can introduce a positive definite Riemannian metric in M^n with $\phi(a_1, a_2, \cdots, a_k)$-structure with respect to which $L_1, L_2, \cdots, L_k$ are mutually orthogonal such that

$$g(X,Y) = g_1(X,Y) + g_2(X,Y) + \cdots + g_k(X,Y), \tag{4.6}$$

for any vector fields X and Y on M^n,where we put

$$g_i(X,Y) = g(l_iX, l_iY), \quad (i = 1, 2, \cdots k). \tag{4.7}$$

For any vector fields X and Y on M^n,we get

$$\begin{aligned} g(\phi l_iX, \phi l_iY) &= g_i(\phi X, \phi Y) = g_i(\sqrt{a_i}\phi_iX, \sqrt{a_i}\phi_iY) \\ &= a_ig_i(\phi_iX, \phi_iY) = a_ig_i(X,Y), \end{aligned}$$

from which

$$g_i(\phi X, \phi Y) = a_ig_i(X,Y), \tag{4.8}$$

$$\begin{aligned} g(\phi X, \phi Y) = a_1g_1(X,Y) + a_2g_2(X,Y) + \cdots + \\ \cdots + a_kg_k(X,Y). \end{aligned} \tag{4.9}$$

From (4.8) we get

$$\begin{aligned} a_ig_i(\phi X, Y) &= g_i(\phi^2X, \phi Y) \\ &= g_i(-a_iX, \phi Y) \\ &= -a_ig_i(X, \phi Y). \end{aligned}$$

Since a_i is a positive real,we have

$$g_i(\phi X, Y) = -g_i(X, \phi Y),$$

from which

$$g(\phi X, Y) = -g(X, \phi Y). \tag{4.10}$$

THEOREM 4.2. *Let M^n be a manifold with $\phi(a_1, a_2, \cdots, a_k)$ - structure. Then there exists a positive definite Riemannian metric g with respect to which $L_1, L_2, \cdots, L_k$ are mutually orthogonal such that*

$$g(\phi X, Y) = -g(X, \phi Y)$$

for any vector fields X and Y on M^n.

Next,take a vector e_i in L_i, then ϕe_i is also in L_i and perpendicular to e_i, and ϕe_i has the same length as $\sqrt{a_i} e_i$ with respect to g. Consequently,we can choose $2r_i(= \dim L_i)$ orthogonal basis $\{e_{i1}, \cdots, e_{i2r_i}\}$ in L_i such that

$$\phi e_{i1} = \sqrt{a}_i e_{ir_i+1}, \cdots, \phi e_{ir_i} = \sqrt{a}_i e_{2r_i}.$$

Then,with respect to the orthonormal basis $dsize\{e_{11}, \cdots, e_{1r_1}, \cdots, e_{k1}, \cdots, e_{k2r_k}\}$ the tensor g and ϕ have the components:

$$g = \begin{pmatrix} E_{r_1} & 0 & \dots & 0 & 0 \\ 0 & E_{r_1} & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & E_{r_k} & 0 \\ 0 & 0 & \dots & 0 & E_{r_k} \end{pmatrix} \tag{4.11}$$

and

$$\begin{pmatrix} 0 & \sqrt{a}_i E_{r_1} & \dots & 0 & 0 \\ -\sqrt{a}_i E_{r_1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \sqrt{a}_k E_{r_k} \\ 0 & 0 & \dots & -\sqrt{a}_k E_{r_k} & 0 \end{pmatrix} \tag{4.12}$$

We call such a frame an adapted frame of $\phi(a_1, a_2, \cdots, a_k)$-structure. Let $\{\overline{e}_i\}$ be another adapted frame in which g and ϕ have the same components as (4.11) and (4.12) respectively. Put

$$\overline{e}_i = T_i^j e_j,$$

then orthogonal matrix T has the form:

$$T = \begin{pmatrix} A_{r_1} & B_{r_1} & \dots & 0 & 0 \\ -B_{r_1} & A_{r_1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C_{r_k} & D_{r_k} \\ 0 & 0 & \dots & -D_{r_k} & C_{r_k} \end{pmatrix} \tag{4.13}$$

Thus the structure group of tangent bundle of the manifold M^n can be reduced to $U(r_1)\times U(r_2)\times\cdots\times U(r_k)$. Conversely,if the structure group of the tangent bundle of the manifold M^n can be reduced to $U(r_1)\times U(r_2)\times\cdots\times U(r_k)$, then we can define a positive definite Riemannian metric g and a $\phi(a_1,a_2,\cdots,a_k)$-structrue with matrices (4.11) and (4.12) with respect to the adapted frame. Then we have

$$\phi^2 = \begin{pmatrix} -a_1E_{r_1} & 0 & \dots & 0 & 0 \\ 0 & -a_1E_{r_1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -a_kE_{r_k} & 0 \\ 0 & 0 & \dots & 0 & -a_kE_{r_k} \end{pmatrix} \tag{4.14}$$

and it is easily verified that $(\phi^2+a_1)(\phi^2+a_2)\cdots(\phi^2+a_k)=0$. Thus we have

THEOREM 4.3. *A necessary and sufficient condition for an n-dimensional manifold to admit a $\phi(a_1,a_2,\cdots,a_k)$-structure is that the group of the tangent bundle be reduced the group $U(r_1)\times U(r_2)\times\cdots U(r_k)$, where $n=2r_1+2r_2+\cdots+2r_k$.*

EXAMPLE. Let CP^{r_i} be the complex projective space of $\dim_C = r_i$. Then $M = CP^{r_1}\times\cdots\times CP^{r_k}$ admits a $\phi(a_1,a_2,\cdots,a_k)$-structure.

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Department of Mathematics Education
Cheju National Teacher's College
Cheju 690-060, Korea