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MANIFOLDS SATISFYING SIMPLE PRODUCT TUBE FORMULAS

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1. Introduction

Let $P \subset M$ be an embedding of a compact p-dimensional manifold P into an n-dimensional Riemannian manifold M. We denote by $V_P^M(r)$ the n-dimensional volume of a solid tube of radius r about P and by $A_P^M(r)$ the $(n-1)$-dimensional volume of its boundary. Throughout this paper we assume that $r > 0$ is less than or equal to the distance from P to its nearest focal point. Then it is well-known that

$$A_P^M(r) = \frac{d}{dr} V_P^M(r). \tag{1}$$

Let $P \subset M$ and $Q \subset N$ be two embeddings, and $P \times Q \subset M \times N$ the corresponding embedding of the product manifold $P \times Q$ to the Riemannian product manifold $M \times N$. The fundamental product formula for the volume of a tube can be written as ([7])

$$A_{P\times Q}^{M\times N}(r) = r \int_0^{\pi/2} A_P^M(r\cos\theta) A_Q^N(r\sin\theta)\, d\theta. \tag{2}$$

From now on we assume that

(3) "$Q \subset N$ is a 0-dimensional submanifold of 2-dimensional locally Euclidean space
or a 1-dimensional submanifold of 3-dimensional locally Euclidean space."

In [7] the second author showed that $Q \subset N$ satisfies

$$A_{P\times Q}^{M\times N}(r) = V_P^M(r) A_Q^N(r) \tag{4}$$

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for any $P \subset M$.

In this paper we characterize some low-dimensional spaces of constant curvature by several product formulas similar to (4). Specifically we consider the product relations (A) $\sim$ (H) with constants a, b, c :

(A) $(2r+a)A_{P\times Q}^{M\times N}(r) = (r^2+ar+b)A_P^M(r)A_Q^N(r)$
(B) $(3r^2+2ar+b)A_{P\times Q}^{M\times N}(r) = (r^3+ar^2+br+c)A_P^M(r)A_Q^N(r)$
(C) $(a\sin ar)A_{P\times Q}^{M\times N}(r) = (b-\cos ar)A_P^M(r)A_Q^N(r)$
(D) $(b+a\cos ar)A_{P\times Q}^{M\times N}(r) = (c+br+\sin ar)A_P^M(r)A_Q^N(r)$
(E) $(r+a)A_{P\times Q}^{M\times N}(r) = (r+2a)A_P^M(r)V_Q^N(r)$
(F) $(r^2+ar+b)A_{P\times Q}^{M\times N}(r) = \left(\frac{2}{3}r^2+ar+2b\right)A_P^M(r)V_Q^N(r)$
(G) $(ar\sin ar)A_{P\times Q}^{M\times N}(r) = 2(b-\cos ar)A_P^M(r)V_Q^N(r)$
(H) $r(b+a\cos ar)A_{P\times Q}^{M\times N}(r) = 2(c+br+\sin ar)A_P^M(r)V_Q^N(r)$.

Then the following theorems show that there are restrictions on the manifold M and on the constants in order that one of (A) $\sim$ (H) holds for $Q \subset N$ when $\dim P = 0$ or 1.

THEOREM 1. *Let $P \subset M$ be an embedding with $\dim P = 0$. Assume that $Q \subset N$ satisfies* (3).

(i) *If $P \subset M$ satisfies* (A) *(resp.* (E)) *for $Q \subset N$, then M is locally Euclidean space of dimension* 2 *and $a = b = 0$ (resp. $a = 0$).*
(ii) *If $P \subset M$ satisfies* (B) *(resp.* (F)) *for $Q \subset N$, then M is locally Euclidean space of dimension* 3 *and $a = b = c = 0$ (resp. $a = b = 0$).*
(iii) *If $P \subset M$ satisfies either* (C) *or* (G) *for $Q \subset N$, then M is a* 2*-dimensional space of constant curvature a^2 and $b = 1$.*
(iv) *If $P \subset M$ satisfies either* (D) *or* (H) *for $Q \subset N$, then M is a* 3*-dimensional space of constant curvature $a^2/4$ and $b = -a$, $c = 0$.*

THEOREM 2. *Let $P \subset M$ be an embedding with $\dim P = 1$. Assume that $Q \subset N$ satisfies* (3).

(i) *If $P \subset M$ satisfies* (A) *(resp.* (E)) *for $Q \subset N$, then M is locally Euclidean space of dimension* 2 *or* 3 *and $a = b = 0$ (resp. $a = 0$).*
(ii) *If $P \subset M$ satisfies* (B) *(resp.* (F)) *for $Q \subset N$, then M is locally Euclidean space of dimension* 2, 3 *or* 4 *and $a = b = c = 0$ (resp.*

$a = b = 0$).

(iii) *If* $P \subset M$ *satisfies either* (C) *or* (G) *for* $Q \subset N$, *then* M *is a 3-dimensional space of constant curvature* $a^2/4$ *and* $b = 1$.

(iv) *If* $P \subset M$ *satisfies either* (D) *or* (H) *for* $Q \subset N$, *then* M *is a 2-dimensional space of constant curvature* a^2 *and* $b = c = 0$.

REMARK. With the usual conventions $\sin it = i \sinh t$, $\cos it = \cosh t$, $t \in \mathbf{R}$, the above theorems also include the cases of constant negative curvature a^2 (when a is pure imaginary).

2. Preliminaries

Before proving the theorems we review a few necessary facts.

From the volume formula for a geodesic ball in non-Euclidean space $\mathbf{E}^n(K)$ of constant curvature K (see for example [2])

$$(5) \qquad A_P^{\mathbf{E}^n(K)}(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \left(\frac{\sin \sqrt{K} r}{\sqrt{K}} \right)^{n-1}, \quad \text{where } P \text{ is a point,}$$

it is not difficult to see that

$$(6) \qquad A''(r) + KA(r) = 0 \quad \text{if} \quad n = \dim \mathbf{E}^n(K) = 2$$

and

$$(7) \qquad A''(r) + 4KA(r) = 0 \quad \text{if} \quad n = 3,$$

where $A(r) = A_P^{\mathbf{E}^n(K)}(r)$.

The function $A(r)$ can be regarded as the *growth function* of tubular hypersurfaces. In [3] Gray and Vanhecke strengthened the result of [5] and prove the following.

THEOREM 3. *Suppose that the growth function* $A(r)$ *of each geodesic sphere satisfies*

$$(8) \qquad A''(r) + c(r)A(r) = 0$$

for small $r > 0$. *Then* M *has constant curvature* $K = c(r)$ *and* $\dim M = 2$.

The result analogous to Theorem 3 is as follows ([6]).

THEOREM 4. *Suppose that the growth function $A(r)$ of each tubular hypersurface about any geodesic segment satisfies (6) for small $r > 0$, then M is a space of constant curvature K of dimension 2 or 3. If $n = \dim M = 2$, then $c(r) = K$; if $n = 3$, then $c(r) = 4K$.*

3. Proof of Theorems

We only prove (ii) in Theorem 1 and (iv) in Theorem 2 since proofs are similar in all cases.

Proof of (ii) in Theorem 1. Let $\dim P = 0$. Suppose that $P \subset M$ satisfies (B) or (F) for any $Q \subset N$. Then (B) (resp. (F)) together with (4) gives

(9)

$$\frac{A_P^M(r)}{V_P^M(r)} = \frac{3r^2 + 2ar + b}{r^3 + ar^2 + br + c} \qquad \left(\text{resp. } \frac{A_P^M(r)}{V_P^M(r)} = \frac{2r^2 + 2ar + 2b}{2r^3/3 + ar^2 + 2br}\right).$$

Integrating (9) with respect to r, we see that

$$V_P^M(r) = \text{const.}(r^3 + ar^2 + br + c).$$

It follows that $\frac{d^3}{dr^3} A_P^M(r) = 0$. Thus by Theorem 13.4 [3, p.196] (see also [1]) M is locally Euclidean space of dimension 3. Furthermore from (5) we should have $a = b = c = 0$ (resp. $a = b = 0$). Finally this $P \subset M$ actually satisfies $3A_{P\times Q}^{M\times N}(r) = rA_P^M(r)A_Q^N(r)$ or $3A_{P\times Q}^{M\times N}(r) = 2A_P^M(r)V_Q^N(r)$.

Proof of (iv) in Theorem 2. Let $\dim P = 1$. If $P \subset M$ satisfies (D) or (H) for any $Q \subset N$, then we have

$$\frac{A_P^M(r)}{V_P^M(r)} = \frac{b + a\cos ar}{c + br + \sin ar}.$$

Hence $A_P^M(r) = A(r)$ satisfies

$$A''(r) + a^2 A(r) = 0.$$

The conclusion of (iv) now follows from Theorem 4 since $P \subset M$ satisfies $a\cos ar A_{P\times Q}^{M\times N}(r) = \sin ar A_P^M(r)A_Q^N(r)$ or $ar\cos ar = 2\sin ar A_P^M(r)V_Q^N(r)$.

References

1. J. Erbacher, *Riemannian manifolds of constant curvature and the growth function of submanifolds*, Michigan Math. J. **19** (1972), 215–223.
2. A. Gray, *The volume of a small geodesic ball of a Riemannian manifold*, Michigan Math. J. **20** (1973), 329–344.
3. A. Gray and L. Vanhecke, *Riemannian geometry as determined by the volumes of small geodesic balls*, Acta Math. **142** (1979), 157–198.
4. ———, *The volumes of tubes about curves in a Riemannian manifold*, Proc. London Math. Soc. **44** (1982), 215–243.
5. R.A. Holzsager and H. Wu, *A characterization of two-dimensional Riemannian manifolds of constant curvature*, Michigan Math. J. **17** (1970), 297–299.
6. S. Lee, *The growth function of tubes about geodesics*, Comm. Korean Math. Soc. **3** (1988), 249–256.
7. ———, *Product tube formulas*, Illinois J. Math. **33** (1989), 153–161.

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