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LEAST SQUARE–COLLOCATION APPROXIMATIONS FOR DIFFUSION EQUATIONS WITH RANDOM INITIAL DATA

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1. Introduction

In this paper we shall treat a linear parabolic equation of the form

$$u_t - \Delta u = f(x,t,\omega), \qquad (x,t,\omega) \in Q \times (0,T] \times \Omega \tag{1.1}$$

subject to the Dirichlet boundary condition

$$u = 0 \qquad (x,t,\omega) \in \partial Q \times (0,T] \times \Omega \tag{1.2}$$

and the initial condition

$$u(x,0,\omega) = u_0(x,\omega), \qquad (x,\omega) \in Q \times \Omega, \tag{1.3}$$

where $Q \equiv I \times I$, $I = (0,1)$, $f(x,t,\omega)$ and $u_0(x,\omega)$ are independent random functions on a complete probability space $(\Omega, \mathcal{F}, P)$, and $\Delta =$ Laplacian operator.

Approximation methods of problem concerning random differential and integral equations can be found in many papers and monographs : Bharucha-Reid [1]; Sun [7]; Tosaka [8], [9]; Larsen [4]; Choi and Kwak [2], [3]. Sun has investigated a finite element method for a stochastic Strum-Liouville problem and Tasaka has studied statistical properties of a finite element approximation to a random heat equation. Very recently Larsen [4] has introduced the stochastic Sobolev spaces and their approximation properties. Choi and Kwak [2] have shown that strong

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convergence properties hold for stochastic finite element approximations to a heat equation with a random initial condition.

The well-known advantage of collocation finite element procedure over Galerkin finite element procedure is that the formation of the coefficients in the equation which determine an approximation is very fast since no integral need be evaluated or approximated.

The idea of collocation at Gaussian points was introduced and analyzed for two dimensional elliptic problems by Percel and Wheeler [11].

In this work we shall utilize the advantage of collocation method to extend the idea and procedure in [2], [3] and obtain probabilistic convergence of the least square-collocation approximations to the problem (1.1)-(1.3).

The outline of this paper is as follows. In section 2 we shall establish our notation and state auxiliary results which will be used to prove random error estimates, and we shall present and prove the main theorems concerning the convergence and rate of convergence for the least square-collocation finite element approximations.

2. Notations and main theorems

Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ be the uniform partition of I with mesh size $h = \frac{1}{N}$ and set $I_i = (x_{i-1}, x_i)$. We denote by M_k^r the finite dimensional space of piecewise polynomials defined by

$$M_k(r,h) = \{v \in C^k(I) : v|_{I_i} \in P_r(I_i),\ 1 \le i \le N\}$$

for $0 \le k \le r-1$ and

$$M_{-1}(r,h) = \{v \in L^2(\overline{I}) : v|_{I_i} \in P_r(I_i),\ 1 \le i \le N\}$$

where $P_r(E)$ denotes the class of polynomials of degree at most $r-1$ on E, $r \ge 3$.

Given two uniform partitions Δ's, let Λ be the collection of grid lines λ in $\overline{Q}$ of the form $\{x_k\} \times I$ or $\{y_\ell\} \times I$, x_k, $y_\ell \in \Delta$.

If X and Y are two spaces of functions on $\overline{I}$, then $X \otimes Y$ is defined to be the space of functions on $\overline{Q}$ consisting of all finite linear combinations of products of the form $f(x)g(x)$, $f \in X$ and $g \in Y$. Let us define

$$M_r(h) = M_k^0(r,h) \otimes M_k^0(r,h)$$

where $M_k^0(r,h) = \{v \in M_k(r,h) : v(0) = v(1) = 0\}$.

Let $\xi_{i,k} = x_{k-1} + h\xi_i$, $1 \le i \le r-1$, $1 \le k \le N$, and $\xi_{j,\ell} = y_{\ell-1} + h\xi_j$, $1 \le j \le r-1$, $1 \le \ell \le N$, where ξ_i's are the simple roots of Legendre polynomial of degree $r-1$. Note that $\xi_{i,k}$'s are the Gaussian points of the kth subinterval (x_{k-1}, x_k). Thus we define the collocation points in $Q \subset \mathbf{R}^2$ by

$$C_r = \{(\xi_{i,k}, \xi_{j,\ell}) \in Q : 1 \le i,j \le r-1,\ 1 \le k,\ell \le N\}.$$

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm for $L^2(Q)$. Denote by $H^k(Q)$, $k = 1, 2, \ldots$, the Sobolev space of functions $v \in L^2(Q)$ such that the weak derivatives

$$D^\alpha v \in L^2(Q), \qquad |\alpha| \le k,$$

and with the norm $\|\cdot\|_k$,

$$\|v\|_k^2 = \sum_{|\alpha| \le k} \|D^\alpha v\|^2, \qquad v \in H^k(Q).$$

We assume that

$$(2.1)\quad f \in L^2((0,T] \times Q \times \Omega) \times C^1((0,T] \times \Omega : C^k(Q)), \quad 0 \le k \le r-1.$$

Denote the expectation of $f(x,t,\cdot)$ and $\|f(\cdot,t,\cdot)\|$ by

$$\langle f(x,t) \rangle = \int_\Omega f(x,t,\omega)\, dP(\omega)$$

and

$$\langle \|f(t)\|^2 \rangle = \int_{Q\times\Omega} |f(x,t,\omega)|^2 \, dx\, dP(\omega).$$

For simplicity we assume that

$$(2.2)\qquad u_0(x,\omega) = 0, \qquad \forall \omega \in \Omega.$$

It is well known that for given $f(x,t,\omega)$ satisfying (2.1) and for each $\omega \in \Omega$, the equation (1.1), (1.2) and (2.2) has a unique solution u. We shall prove the existence and uniqueness of the least square-collocation finite element approximation $U^h \in M_r(h)$, defined by $U^h : [0,T] \to M_r(h)$

$$(2.3)\qquad \begin{aligned} U_t^h - \Delta U^h &= f \quad \text{on} \quad C_r \\ U^h(0) &= 0 \quad \text{at} \quad t = 0 \end{aligned}$$

THEOREM 2.1. *If for each $\omega \in \Omega$, $f(x,t,\omega)$ belongs to $C^1((0,T] : C^k(Q)) \cap L^2((0,T) \times Q)$, then there exists a unique solution $U^h \in M_r(h)$ which satisfies (2.3).*

Proof. For each $\omega \in \Omega$, the equation (2.3) is deterministic, so by the lemma in [11], the assertion follows. We refer to Wheeler [11] for details.

LEMMA 2.1. *If $f \in L^2((0,T] \times Q \times \Omega)$, then $\forall t \in [0,T]$,*

$$\langle \|u(t)\|^2 \rangle + \left\langle \int_0^T \|u(s)\|_1^2 \, dx \right\rangle \leq C \left\langle \int_0^T \|f(x)\|^2 \, ds \right\rangle .$$

LEMMA 2.2. *If f, f_t, $f_{tt} \in L^2((0,T) \times Q \times \Omega)$, then $\forall t \in [0,T]$.*

$$\langle \|u_t(t)\|^2 \rangle + \left\langle \int_0^T \|u_s(s)\|_1^2 \, dt \right\rangle \leq C \left\langle \int_0^T \|f_s(s)\|^2 \, ds \right\rangle$$

$$\langle \|u_{tt}(t)\|^2 \rangle + \left\langle \int_0^T \|u_{ss}(s)\|_1^2 \, dx \right\rangle \leq C \left\langle \int_0^T \|f_{ss}(s)\|^2 \, ds \right\rangle .$$

The proofs of the above two lemmas are straightforward by the usual estimates procedure. so we omit it.

The following theorem is an extension of the result in [11]. We state this without proof.

THEOREM A. *Let u and U be the solutions of (1.1)–(1.3) and (2.3) respectively. If $u \in H^{r+3}(Q)$, $U \in M_r(h)$ and $f \in L^2([0,T] \times Q \times \Omega)$, then for each $t \in (0,T]$*

$$\|u - U\|_1 \leq Ch^r[h\|u\|_{r+3} + \|u\|_{r+2}].$$

For a sequence of discretization parameter $h_j \in (0, \frac{1}{2}]$, $j = 1, 2, \ldots$, with $h_j \downarrow 0$ as $j \to \infty$, $M_r(h_j) \subset H_0^{r+2}(Q)$ is a finite dimensional subspace such that for all $v \in H_0^{r+2}(Q) \cap H^m(Q)$,

$$\inf_{\phi \in M_r(h_j)} \|v - \phi\|_\ell \leq Ch_j^{m-\ell}\|v\|_m, \qquad 0 \leq \ell < m,$$

where the constant C does not depend on h_j or v.

We now state one of our main result and prove it.

THEOREM 2.2. *Suppose that* $\sum_{j=1}^{\infty} h_j^{\nu}$, *for* $0 < \nu < r-1$, *and* f, f_t, $f_{tt} \in L^2((0,T] \times Q \times \Omega) \cap L^\infty((0,T] \times Q \times \Omega)$, *then, for each* $t \in (0,T]$,

$$\|u - U^{h_j}\|_1 = O(h_j^{\nu}) \qquad P - a.s., \qquad \nu > 0.$$

Proof. Following the argument in [2], since

$$P\left(\|u - U^{h_j}\|_1 \geq h_j^{\nu}\right) \leq \frac{\langle \|u - U^{h_j}\|_1\|^2 \rangle}{h_j^{2\nu}}$$

$$\sum_{j=1}^{\infty} P(\|u - U^{h_j}\|_1 \geq h_j^{\nu}) \leq \sum_{j=1}^{\infty} \frac{\langle \|u - U^{h_j}\|_1^2 \rangle}{h_j^{2\nu}}.$$

Using the Theorem A, Lemma 2.1 and the Lemma 2.2. We obtain

$$\sum_{j=1}^{\infty} P(\|u - U^{h_j}\|_1 \geq h_j^{\nu})$$

$$\leq C \left\{ \left\langle \int_0^T [\|f(s)\|^2 + \|f_s(s)\|^2 + \|f_{ss}(s)\|^2]\, ds \right\rangle \right\} \times \sum_{j=1}^{\infty} h_j^{2(r-\nu)} < \infty,$$
$$0 < \nu < r-1.$$

Thus the Borel-Cantelli's Lemma implies the assertion.

REMARK. The result above is in fact an application of the theorem in [2] to the least square-collocation finite element approximation. In this case the rate of convergence ν depends on the degree $r-1$ of piecewise polynomials, which is a great improvement of the convergent rate comparing with the case in [2].

Now we assume that the data values $\underset{\sim}{f} = (f_1, f_2, \cdots, f_M)^T$ are sampled at the collocation points $(\xi_{i,k}, \xi_{j,\ell})$, $1 \leq i,j \leq r$, $1 \leq k, \ell \leq N$, where M is the number of collocation points $(\xi_{i,k}, \xi_{j,\ell})$. Then $\underset{\sim}{f}$ is a random vector whose entry f_j's are realization of a sample event $\omega \in \Omega$.

Let $\{h_{j_\alpha}\}_{\alpha=1}^{N}$ be a random finite subsequence of $\{h_j\}_{j=1}^{\infty}$ with $\sum_{i=1}^{\infty} h_j^{\nu} < \infty$, $0 < \nu < r$. We call $\{h_{j_\alpha}\}_{\alpha=1}^{N}$ the sample meshes of size N from $\{h_j\}_{j=1}^{\infty}$.

DEFINITION. Define the least square-collocation sample mean $\overline{U}_N$ by

$$\overline{U}_N = \frac{1}{N}\sum_{\alpha=1}^{N} U^{h_{j_\alpha}},$$

where each $U^{h_{j_\alpha}}$ is the solution of (2.3) with h_{j_α}. Note that $\overline{U}_N$ can be considered as a statistical estimator for the exact solution u.

The following theorem is an analogue of one in our previous work [2], [3].

THEOREM. *Assume the hypotheses in the Theorem 2.2. Let* $\{h_{j_\alpha}\}_{\alpha=1}^{N}$ *be a random finite subsequence as above. Then we have, for each* $t \in (0,T]$,

$$\|u - \overline{U}_N\|_1 = O(h_j^\nu), \qquad 0 < \nu < r-1$$

Proof. We start with

$$\|u - \overline{U}_N\|_1^2 = \left\| \frac{1}{N}\sum_{\alpha=1}^{N}(u - U^{h_{j_\alpha}})\right\|_1^2 \le \frac{2}{N^2}\sum_{\alpha=1}^{N}\|u - U^{h_{j_\alpha}}\|_1^2.$$

Take expectation on both sides,

$$\langle \|u - \overline{U}_N\|_1^2\rangle \le \frac{2}{N^2}\sum_{\alpha=1}^{N}\langle \|u - U^{h_{j_\alpha}}\|_1\rangle \le C\sum_{j=1}^{\infty} h_j^{2r+2}.$$

Thus

$$P\left(\|u - \overline{U}_N\|_1 \ge h_{j_\alpha}^\nu\right) \le C\sum_{j=1}^{\infty} h_j^{2(r-\nu)} < \infty,$$

$0 < \nu < r-1$, which implies the assertion by the Borel-Cantelli's lemma.

COROLLARY 1. *Let $\langle u(x,t)\rangle$ be the expectation of the exact random solution of (1.1)-(1.3). Then under the assumptions of the Theorem 2.2, $\overline{U}_N$ converges in probability measure to $\langle u(x,t)\rangle$ with $\|\cdot\|_1$ norm.*

Proof. By the triangle inequality, $\forall\omega\in\Omega$,

$$\|\overline{U}_N - \langle u\rangle\|_1^2 \le \|\overline{U}_N - u + u - \langle u\rangle\|_1^2$$

since $\forall\omega\in\Omega$,

$$\begin{aligned}\|\overline{U}_N - u + u - \langle u\rangle\|_1^2 &= \left\|\frac{1}{N}\sum_{\alpha=1}^{N}\{[U^{h_{j\alpha}} - u] + [u - \langle u\rangle]\}\right\|_1^2 \\ &\le \frac{1}{N^2}\sum_{\alpha=1}^{N}\|u - U^{h_{j\alpha}}\|_1 + \frac{1}{N}\|u - \langle u\rangle\|_1^2 \to 0\end{aligned}$$

as $N\uparrow\infty$, the result follows.

COROLLARY 2. *Let N_j's be the subsample size corresponding to h_j's, $N=\sum N_j$, such that*

$$\sum N_j^{-1}h_j^{2\nu} < \infty, \qquad (0<\nu<r-1).$$

Then $\|\overline{U}_N - \langle u\rangle\|_1 = O(h_j^{\nu})$ $P-a.s.$.

Proof. Observe that

$$\begin{aligned}P\left(\|\overline{U}_N - \langle u\rangle\|_1 \ge h_j^{\nu}\right) &\le \sum_{j=1}^{N}\frac{\langle\|\overline{U}_N - \langle u\rangle\|_1^2\rangle}{h_j^{2\nu}} \\ &\le \sum_{j=1}^{\infty}\frac{1}{h_j^{2\nu}}\left\{\frac{C}{N^2}\sum_{j=1}^{\infty}h_j^{2r+2} + \frac{2}{N}\langle\|u - \langle u\rangle\|\rangle_1\right\} \\ &\le C\sum_{j=1}^{\infty}\frac{1}{Nh_j^{2\nu}} \le C\sum N_j^{-1}h_j^{2\nu} < \infty,\end{aligned}$$

from which the assertion follows.

REMARK. The above two corollaries show that the same line of stochastic convergent properties hold for the stochastic collocation approximation.

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