Comm. Korean Math. Soc. 6(1991), No. 1, pp. 31–38

CHAIN RECURRENCE AND RESIDUAL SETS

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M. Hurley[3] showed that there is a residual subset J of the set of C^1-diffeomorphisms on any compact Riemannian manifold M such that for any $f \in J$, the maps

(i) $f \longrightarrow CR(f)$
(ii) $f \longrightarrow$ (*number of chain components forf*)

are continuous.

The purpose of theis paper is to extend the above result to C^1-flows on a compact Riemannian manifold M. The main result is

THEOREM. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}'(M)$ such that for any $X \in \mathcal{X}'(M)$ the maps*

(i) $X \longrightarrow CR(X)$
(ii) $X \longrightarrow$ (*number of chain components for* X)

are continuous.

Throughout this paper M will denote a compact Riemannian manifold with metric d, and $\mathcal{X}'$ will denote the set of C^1 -flows on M, that is,

$$\begin{aligned}\mathcal{X}'(M) =&\{X : M \times \mathbf{R} \longrightarrow M \text{ is } C^1 : X(x,0) = x,\\ &X(x,s+t) = X(X(x,s),t),\ x \in M,\ s,t \in \mathbf{R}\}.\end{aligned}$$

We shall denote $X(x,t)$ by $X^t(x)$ or xt for $(x,t) \in M \times \mathbf{R}$. We give $\mathcal{X}'(M)$ the topology generated by the C^0-metric on generating vector fields

$$d_0(X,Y) = \sup\{\|X'(x,0) - Y'(x,0)\| : X \in M\}.$$

Received Aug 29, 1990.
The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1989.

Let $\epsilon > 0$ and $a > 0$ be given. For any $x, y \in M$, we say that a sequence $\{(x_i, t_i)\}_{i=1}^{n}$ in $M \times \mathbf{R}$ is called an (ϵ, a)-chain going from x to y for $X \in \mathcal{X}(M)$ if

$$\text{(i) } x_1 = x,\ t_i \geq a,\ i = 1, \cdots, n,$$
$$\text{(ii) } d(x_{i-1} - t_{i-1}, x_i) < \epsilon \text{ and } d(x_n t_n, y) < \epsilon,$$

$i = 2, \cdots, n$. The sequence $\{(x_i, t_i)\}_{i=1}^{n}$ is called an ϵ-chain if $1 \leq t_i \leq 2$ for each $i = l, \cdots, n$.

The chain recurrent set for X, denoted by $CR(X)$, is the set $\{x \in M$: for each $\epsilon > 0$ and $a > 0$ there is an (ϵ, a)-chain going from x to $x\}$. There is a natural equivalence relation defined on $CR(X) : x \sim y$ if and only if for each $\epsilon > 0$ and $a > 0$ there exist two (ϵ, a)-chains; going from x to y and going from y to x . Equivalence classes under this equivalence relation are called chain components of X. It turns out for flows on compact manifolds that the chain components are exactly the connected components of $CR(X)$ (See [4]).

To Prove our theorem, we need some lemmas.

LEMMA 1. *Let $h : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ be a strictly positive (but not necessarily continuous) function such that*

(1) *for any $\epsilon > 0$, $d(x, y) < h(\epsilon)$ and $1 \leq t \leq 2$ imply that $d(xt, yt) < \epsilon/4$.*
(2) *for any $i < j$, $h^j < h^i$.*

If $\{(x_i, t_i)\}_{i=1}^{n}$ is an $h^n(\epsilon)$-chain, then $d(x_1 \sum_{i=1}^{n} t_i, x_n t_n) < \epsilon/2$.

Proof. Let $\epsilon > 0$ be arbitrary. Then we can choose $0 < \delta(\epsilon) < \epsilon$ such that if $d(x, y) < \delta$ then $d(xt, yt) < \epsilon/4$ for any $t \in [1, 2]$. The function $h : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ defined by $h(\epsilon) = \delta(\epsilon)$ satisfies the conditions (i) and (ii). Hence we amy assume that the function h exists.

We will prove the lemma by induction on n. If $n = 1$, there is nothing to prove. If $n = 2$, then the result is clear by the condition (i). Suppose the result is known for a given $n \geq 2$; we shall establish it for $n + 1$.

Let $\{(x_i,t_i)\}_{i=1}^{n+1}$ be an $h^{n+1}(\epsilon)$-chain. Since $\{(x_i,t_i)\}_{i=1}^{n}$ is an $h^n(h(\epsilon))$-chain, we have

$$d(x_1\sum_{i=1}^{n} t_i, x_n t_n) < h(\epsilon)/2 < h(\epsilon).$$

By the assumption (i), we get

$$d(x_1\sum_{i=1}^{n+1} t_i, x_n(t_n + t_{n+1})) < \epsilon/4.$$

By the definition of the chain, we obtain

$$d(x_n t_n, x_{n+1}) < h^{n+1}(\epsilon) < h(\epsilon),$$

and so

$$d(x_n(t_n+t_{n+1}), x_{n+1}t_{n+1}) < \epsilon/4.$$

Consequently we have

$$d(x_1\sum_{i=1}^{n+1} t_i, x_{n+1}t_{n+1}) < \epsilon/2.$$

LEMMA 2. *For any tow points x,y in $CR(M)$, $x \sim y$ if and only if for any $\epsilon > 0$ there exist two ϵ-chains; going from x to y and going from y to x.*

Proof. The necessity is obvious. To prove the sufficiency, we let $\epsilon > 0$ and $a > 0$ be arbitrary. We may assume that $a > 1$. We shall construct (ϵ, a)-cahins; going from x to y and going from y to x. Let p be the greatest integer smaller than $a+1$. Choose two $2^{2p}(\epsilon)$-chains; $\{(x_i,t_i)\}_{i=1}^{n}$ going from x to y and $\{(y_j,s_j)\}_{j=1}^{m}$ going from y to x. For each $k \in \mathbf{Z}$ with $1 \le k \le (p+1)n + pm$, define $z_k \in M$ and $1 \le r_k \le 2$ as follows:

$$z_k = \begin{cases} x_{k-i(n+m)}, & \text{if } i(n+m) < k \le (i+1)n + im, \text{ where } i = 0,1,\cdots,p, \\ y_{k-(j+1)n-jm}, & \text{if } (j+1)n + jm < k \le (j+1)(n+m), \\ & \quad \text{where } j = 0,1,\cdots,p-1, \end{cases}$$

$$r_k = \begin{cases} t_{k-i(n+m)}, & \text{if } i(n+m) < k \le (i+1)n + im, \text{ where } i = 0,1,\cdots,p, \\ s_{k-(j+1)n-jm}, & \text{if } (j+1)n + jm < k \le (j+1)(n+m), \\ & \quad \text{where } j = 0,1,\cdots,p-1. \end{cases}$$

Then $\{(z_l, r_1), \cdots, (z_{(p+1)n+pm}, r_{(p+1)n+pm})\}$ is also an $h^{2p}(\epsilon)$-chain going from x to y. Let $(p+1)n+pm = sp+q$ for some $0 \leq q < p$. Since $h^{2p}(\epsilon) < h^p(\epsilon)$, $\{(z_{(i-1)p+1}, r_{(i-1)p+1}, \cdots, (z_{ip}, r_{ip})\}$ is an $h^p(\epsilon)$-chain. By Lemma 1, we have

$$d(z_{(i-1)p+1} \sum_{j=1}^{p} r_{(i-1)p+j}, z_{ip} r_{ip}) < \epsilon/2.$$

Since $h^{2p}(\epsilon) < h^{p+q}(\epsilon)$, $\{(z_{(s-1)p+1}, r_{(s-1)p+1}), \cdots, (z_{sp+q}, r_{sp+q})\}$ is an $h^{p+q}(\epsilon)$-chain. By Lemma 1, we have

$$d(z_{(s-1)p+1} \sum_{j=1}^{p+q} r_{(s-1)p+j}, z_{sp+q} r_{sp+q}) < \epsilon/2.$$

Let $w_1 = z_1$, $w_2 = z_{p+1}, \cdots, w_s = z_{(s-1)+1}$. Let $\tau_1 = \sum_{j=1}^{p} r_j$, $\tau_2 = \sum_{j=1}^{p} r_{p+j}$, $\cdots$, $\tau_s = \sum_{j=1}^{p+q} r_{(s-1)p+j}$. Then the sequence $\{(w_i, \tau_i)\}_{i=1}^{s}$ is a desired (ϵ, a)-chain going from x to y. Similarly we can construct an (ϵ, a)-chain going from y to x. This completes the proof.

Let X_1 and X_2 be metric spaces with X_2 compact. Let FX_2 be the set of all closed non-empty subsets of X_2 with the Hausdorff metric

$$\rho(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

A map $f : X_1 \longrightarrow FX_2$ is called upper (lower) semicontinuous at $x \in X_1$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d(s, y) < \delta$ then $f(y) \subset B(f(x), \epsilon)$, $(f(y) \subset B(f(x), \epsilon))$ respectively. Recall that a subset S of a topological space X_1 is residual if S can be realized as a countable intersection of open dense subsets of X_1.

LEMMA 3. [3]. *Let $f : X_1 \longrightarrow FX_2$ be upper (lower) semicontinuous. Then the set of all continuity points of f is a residual subset of X_1.*

Now we consider a map

$$CR : \mathcal{X}'(M) \longrightarrow FM$$

sending $X \in \S'(M)$ to $CR(X)$. We will show that there exists a residual subset $\mathcal{R}_1$ of $\mathcal{X}'(M)$ such that the map CR is continuous in each point of $\mathcal{R}$. For this, we need a lemma.

LEMMA 4. *Suppose a sequence X_n in $\mathcal{X}'(M)$ converges to X, $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in M, and that for each n and each positive ϵ there is an ϵ-chain for X_n going from x_n to y_n. Then for each $\epsilon > 0$ there is an ϵ-chain for X going from x to y.*

Proof. Let $\epsilon > 0$ be arbitrary and let $a \in M$. For any $t \in [1,2]$ there is $\delta(a,t) > 0$ such that if $d_0(X,Y) < \delta(a,t), d(a,b) < \delta(a,t)$ and $|t-s| < \delta(a,t)$ then $d(X^t(a), Y^s(b)) < \epsilon/4$. Since [1,2] is compact, there exists a finite set $\{t_i \in [1,2] : i = 1, \cdots, n\}$ such that $[1,2] \subset \cup_{i=1}^n B(t_i, \delta(a,t_i))$. Let $\delta(a) = \min\{\delta(a,t_i) := 1, \cdots, n\}$. Then we have that if $d_0(X,Y) < \delta(a)$ and $d(a,b) < \delta(a)$, then $d(X^t(a), Y^t(b)) < \epsilon/2$ for any $t \in [1,2]$. Hence, for any $a \in M$, we can choose $\delta(a) > 0$ such that if $d_0(X,Y) < \delta(a)$ and $d(a,b) < \delta(a)$, then $d(X^t(a), Y^t(b)) < \epsilon/2$ for any $t \in [1,2]$. Consider the open covering $\{B(a, \delta(a)/2) : a \in M\}$ of M. We can select a finite set $\{a_i \in M : i = 1, \cdots, n\}$ of M such that $M = \bigcup_{i=1}^{n} B(a_i, \delta(a_i)/2)$. Let $\delta = \min\{\delta(a_i)/2 > 0 : i = 1, \cdots, n\}$. Then we have that if $d_0(X,Y) < \delta$ and $d(a,b) < \delta$, then $d(X^t(a), Y^t(b)) < \epsilon$ for any $a, b \in M$ and any $t \in [1,2]$. We may assume that there exists $0 < \delta < \epsilon/3$ such that if $d_0(X,Y), \delta$ and $d(a,b) < \delta$, then $d(X^t(a), Y^t(b)) < \epsilon/3$ for any $t \in [1,2]$. Choose m such that $d_0(X_m, X) < \epsilon, d(x_m, x) < \delta$ and $d(y_m, y) < \delta$. By the assumption, we select $\epsilon/3$-chain $\{(z_i,t_i)\}_{i=1}^n$ for X_m going from x_m to y_m. Then the sequence $\{(x,t_1), (z_2,t_2), \cdots, (z_n,t_n)\}$ in an ϵ-chain for X going from x to y. This completes the proof.

THEOREM 5. *There is a residual subset $\mathcal{R}_1$ of $\mathcal{X}'(M)$ such that the map CR is continuous in each point of $\mathcal{R}_1$.*

Proof. It is enough to show that the map CR is upper semicontinuous. Suppose not. Then we can choose $\epsilon > 0$ such that for each $n > 0$ there exists $X_n \in \mathcal{X}'(M)$ such that

$$d_0(X_n, X) < 1/n \text{ and } CR(X_n) \not\subset B(CR(X), \epsilon).$$

Thus we can select $x_n \in CR(X_n)$ satisfying $d(x_n, CR(X)) \geq \epsilon$. We may assume that $x_n \longrightarrow x \in M$. Then we have $d(x, CR(X)) \geq \epsilon$. Since $X_n \longrightarrow X$ and $x_n \in CR(X_n)$, by lemma 4, we get $x \in CR(X)$. This is a contraction, and so completes the proof.

We consider a map

$$N : \mathcal{X}'(M) \longrightarrow [0, \infty]$$

sending each $X \in \mathcal{X}'(M)$ to the number of chain components for X. We will show that there exists a residual subset $\mathcal{R}$ of $\mathcal{X}'(M)$ such that the map N is continuous in each point of $\mathcal{R}$. For this we need a lemma which characterize the lower semicontinuity using the concept of openness.

LEMMA 6. *A map $f : X_1 \longrightarrow FX_2$ is lower semicontinuous at $x \in X_1$ if and only if for any open subset U of X_2 with $U \cap f(x) \neq \phi$, there exists a neighborhood V of x in X_1 such that $V \cap f(y) \neq \phi$ for any $y \in V$.*

Proof. Suppose that the map f is lower semicontinuous at $x \in X_1$. Let $z \in U \cap f(x)$. Then there is $\epsilon > 0$ satisfying $B(z, \epsilon) \subset U$. By the assumption, there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $f(x) \subset B(f(y), \epsilon)$. Put $V = B(x, \delta)$. For any $y \in V$, we have $z \in f(x) \subset B(f(y), \epsilon)$, and so $d(z, w) < \epsilon$ for some $w \in f(y)$. This means that $f(y) \cap U \neq \phi$.

Let $\epsilon > 0$ be arbitrary. Since $\{B(z, \epsilon/2) : z \in f(x)\}$ is an open covering of $f(x)$, there are $z_1, \cdots, z_n \in f(x)$ such that $f(x) \subset \cup_{i=1}^{n} B(z_i, \epsilon/2)$. By the assumption, there exist neighborhoods V_i of x such that if $y \in V_i$ then $B(z_i, \epsilon/2) \cap f(y) \neq \phi$. Choose $\delta > 0$ satisfying $B(x, \delta) \subset \cap_{i=1}^{n} V_i$. Then we have that if $d(x, y) < \delta$ then $f(x) \subset B(f(y), \epsilon)$. To show this, we let $w \in f(x)$. Then we have $w \in B(z_i, \epsilon/2)$ for some i. Since $y \in B(x, \delta) \subset V_i$, $B(z_i, \epsilon/2) \cap f(y) \neq \phi$, say $u \in B(z_i, \epsilon/2) \cap f(y)$. Then we get

$$d(w, u) \leq d(w, z_i) + f(z_i, u) < \epsilon$$

This implies that $w \in B(f(y), \epsilon)$, and so complets the proof.

THEOREM 7. *There is a residual subset $\mathcal{R}$ of $\mathcal{R}_1$ such that the map N is continuous in each point of $\mathcal{R}$.*

Proof. It is enough to show that the map $N : \mathcal{R}_1 \longrightarrow [0, \infty]$ is lower semicontinuous. Let $X \in \mathcal{R}_1$ and suppose that $N(X)$ is finite so that we can list the X-chain components $M_1, \cdots, M_k$. Then there is $\epsilon > 0$ such that $B(M_i, \epsilon) \cap B(M_j, \epsilon) = \phi$ if $i \neq j$. Since the map CR is continuous, by Lemma 6, we can choose $\delta > 0$ such that if $d_0(X, Y) < \delta$, then

$$CR(Y) \subset B(CR(X), \epsilon) \text{ and } CR(Y) \cap B(M_i, \epsilon) \neq \phi$$

for all $i = 1, \cdots, k$. Let A be a chain component of $CR(Y)$. Then we have

$$A \subset CR(Y) \subset B(CR(X), \epsilon) \subset \cup_{i=1}^{k} B(M_i, \epsilon)$$

Thus we obtain

$$A = A \cap (\cup_{i=1}^{k} B(M_i, \epsilon)) = \cup_{i=1}^{k} (A \cap B(M_i, \epsilon)).$$

Since A is a connected component in $CR(Y)$, there exists a unique i satisfying $A \cap B(M_i, \epsilon) \neq \phi$. Hence we have $A \subset B(M_i, \epsilon)$. For any $i = 1, \cdots, k$, $CR(Y) \cap B(M_i, \epsilon) \neq \phi$. By the above property, we can choose a unique chain component B of $CR(Y)$ such that $B \subset B(M_i, \epsilon)$. This implies that $N(X) \leq N(Y)$.

Suppose that $N(X) = \infty$. For each $n \in \mathbf{Z}^+$, we can choose $\delta(n) > 0$ such that if $d_0(X, Y) < \delta$ the $N(Y) \geq n$, by the above property. This means that $\lim_{Y \longrightarrow X} N(Y) = \infty$. Hence N is continuous at X. This completes the proof.

A residual subset of a residual set is also residual. By combining Theorem 6 and Theroem 7, we obtain the main theorem of this paper.

THEOREM 8. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}'(M)$ such that the maps CR and N are continuous in each point of $\mathcal{R}$.*

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