Comm. Korean Math. Soc. 6(1991), No. 1, pp. 19–26

CHANGE OF SCALE FORMULAS FOR YEH–WIENER INTEGRALS

IL YOO AND GOOG-JOONG YOON

1. Introductory Preliminaris

R.H. Cameron and D.A. Storvick's research [6] on the relationship between Wiener and Feynman integrals has led them to consider the problem of change of scale in Wiener integrals. In [5], they found change of scale formulas for Wiener integrals for a large class of functionals $S(L_2^\nu[a,b])$ wich was defined in [4]. Recently the first author [18] introduced the relationship between the Yeh-Wiener integral and the analytic Yeh-Feynman integral, and obtained change of scale formulas for Yeh-Wiener integrals which we now briefly review.

Let $C_2 = C_2(Q)$ denote Yeh-Wiener space, that is, the space of continuous functions x on $Q = [a,b]\times[c,d]$ such that $x(a,t) = x(s,c) = 0$ for all $(s,t) \in Q$ and let $C_2^\nu = \times_1^\nu C_2(Q)$. We shall say that two functionals $F(\vec{x})$ and $G(\vec{x})$ are equal s-almost everywhere ($s-a.e.$) if for each $\rho > 0$ the equation $F(\rho\vec{x}) = G(\rho\vec{x}))$ holds for almost all $\vec{x} \in C_2^\nu$. For a rather detailed discussion of scale-invariant measurability and its relation with other topics, see [7,13].

Let $M = M(L_2^\nu(Q))$ be the class of complex measures of finite variation defined on $B(L_2^\nu)$, the Borel class of $L_2^\nu(Q)$. The Banach algebra $S(L_2^\nu(Q))$ consists of all functionals F on C_2^ν expressible in the form

$$F(\vec{x}) = \int_{L_2^\nu} \exp\{i\sum_{k=1}^{\nu}\int_Q v_k(s,t)\widetilde{dx}_k(s,t)\}d\mu(\vec{v}) \tag{1.1}$$

for s-a.e. $\vec{x} \in C_2^\nu$ and for some $\mu \in M$ where $\int_Q v_k(s,t)\widetilde{dx}_k(s,t)$ means the Paley-Wiener-Zygmund integral [1, 4, 8, 14, 17]. We note that the

Received June 19,1990. Revised January 9, 1991 .
This study was supported by Yonsei University Faculty Research Grant for 1989.

correspondence $\mu \to F$ is one-one and carries convolution into pointwise multiplication. Moreover the analytic Yeh-Feynman integral exists for every F in $S(L_2^\nu(Q))$ and is given by the formula

$$\int_{C_2^\nu}^{\mathrm{any}\, f_q} F(\vec{x})dx = \int_{L_2^\nu} \exp\left\{ \frac{1}{2qi} \sum_{k=1}^{\nu} \|v_k\|_2^2 \right\} d\mu(\vec{v}). \tag{1.2}$$

In order to state change of scale formulas for Yeh-Wiener integrals, we introduce the piecewise linear functions of two variables as approximators for $\vec{x} \in C_2^\nu(Q)$.

Let l and m be nonnegative integers and consider the division σ of $Q = [a,b] \times [c,d]$ into subrectangles by means of the partition

$$\sigma : a = s_0 < s_1 < \cdots < s_l = b, c = t_0 < t_1 < \cdots < t_m = d$$

For each $\vec{x} = (x^1, \cdots, x^\nu) \in C_2^\nu(Q)$, we define the quadratic approximation $\vec{x}_\sigma = (x_\sigma^1, \cdots, x_\sigma^\nu)$ of $\vec{x}$ based on σ by the formula

(1.3)

$$\begin{aligned} x_\sigma^i(s,t) = {} & \frac{x^i(s_j,t_k) - x^i(s_{j-1},t_k) - x^i(s_j,t_{k-1}) + x^i(s_{j-1},t_{k-1})}{(s_j - s_{j-1})(t_k - t_{k-1})} \\ & \times (s - s_{j-1})(t - t_{k-1}) \\ & + \frac{x^i(s_j,t_{k-1}) - x^i(s_{j-1},t_{k-1})}{s_j - s_{j-1}}(s - s_{j-1}) \\ & + \frac{x^i(s_{j-1},t_k) - x^i(s_{j-1},t_{k-1})}{t_k - t_{k-1}}(t - t_{k-1}) \\ & + x^i(s_{j-1},t_{k-1}) \end{aligned}$$

for $(s,t) \in [s_{j-1}, s_j] \times [t_{k-1}, t_k]$ for $j = 1, \cdots, l, k = 1, \cdots, m$, and $i = l, \cdots, \nu$.

As mentioned before, we now introduce change of scale formulas for Yeh-Wiener integrals of elements in the Banach algebra $S(L_2^\nu(Q))$ [18].

THEOREM 1.1. *Let $\rho > 0$ and let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that the norm $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $l_n m_n$ be the* *number of subrectangles in* σ_n, *Then if* $F \in S(L_2^\nu(Q))$.

(1.4)

$$\int_{C_2^\nu} F(\rho\vec{x})d\vec{x}$$

$$= \lim_{n\to\infty} \rho^{-\nu l_n m_n} \int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\int_Q \left\|\frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s \partial t}\right\|^2 dsdt\right\} F(\vec{x})d\vec{x}$$

THEOREM 1.2. *Let* $\rho > 0$ *and let* $\{\psi_n\}$ *be a complete orthonormal sequence of functions on* Q. *Then if* $F \in S(L_2^\nu(Q))$,

(1.5)

$$\int_{C_2^\nu} F(\rho\vec{x})\widetilde{dx}$$

$$= \lim_{n\to\infty} \rho^{-\nu n} \int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{\nu}\sum_{j=1}^{n}\left[\int_Q \psi_j(s,t)\widetilde{dx}_k(s.t)\right]^2\right\} F(\vec{x})d\vec{x}$$

2. Formulas for Yeh-Wiener Integrals

In this section, we show that the Banach algebra $S(L_2^\nu(Q))$ of analytic Yeh-Feynman integrable functionals is not closed under the uniform convergence, and that change of scale formulas (1.4) and (1.5) for $S(L_2^\nu(Q))$ can be extended to the closure of $S(L_2^\nu(Q))$ under the uniform convergence scale-invariant almost everywhere.

PROPOSITION 2.1. $S(L_2^\nu(Q))$ *is not closed under the uniform convergence.*

Proof. Let $M(\mathcal{R}^\nu)$ be the set of $\mathcal{C}$-valued countably additive Borel Measures on R^ν, and let $\hat{M}(\mathcal{R}^\nu)$ be the set of Fourier transforms of all elements in $M(\mathcal{R}^\nu)$. By a paper of Hewitt [11], there exists a sequence ψ_n of elements in $\hat{M}(\mathrm{R}^\nu)$ such that $\psi_n \to \psi$ uniformly, but $\psi \notin \hat{M}(\mathrm{R}^\nu)$. Now we define the functionals F_n and F from $C_2^\nu(Q)$ to C by

$$F_n(\vec{x}) = \psi_n(\vec{x}(b,d)) \text{ and } F(\vec{x}) = \psi(\vec{x}(b,d)).$$

Then from Theorem 3.1 in [15] it follows that $F_n \in S(L_2^\nu(Q))$ for $n = 1, 2, \cdots$, but $F \notin S(L_2^\nu(Q))$. And also the fact that $\psi_n \to \psi$ uniformly implies $F_n \to F$ uniformly. Thus $S(L_2^\nu(Q))$ is not closed under the uniform convergence.

NOTATION. We shall denote the closure of $S(L_2^\nu(Q))$ under the uniform convergence s-a.e. by $\bar{S}^u(L_2^\nu(Q))$.

PROPOSITION 2.2. *Let $\rho > 0$, let $\{\sigma_n\}$ by a sequence of subdivisions of Q such that the norm $\|\sigma_n\| \to 0$ an $n \to \infty$, and let $l_n m_n$ be the number of sub-rectangles in σ_n. Let Γ be the set of functionals F defined $s - a.e.$ on C_2^ν such that F is bounded s-a.e. on C_2^ν and such that the equation (1.4) holds for F in the sense that both members exist and they are equal. Then Γ is closed with respect to uniform convergence s-a.e. on C_2^ν*

Proof. Let F_q be a sequence of elements in Γ which converges to F uniformly $s - a.e.$ on C_2^ν. Then there exist a positive number B and a subset Ω of C_2^ν with contains s-almost all of C_2^ν such that for all $\vec{x} \in \Omega$

$$|F_q(\vec{x})| \le B \text{ and } |F(\vec{x})| \le B, \tag{2.1}$$

and hence we obtain

$$\int_{C_2^\nu} F(\rho\vec{x})d\vec{x} = \lim_{q\to\infty} \int_{C_2^\nu} F_q(\rho\vec{x})d\vec{x} \tag{2.2}$$

By Proposition 2.3 in [18], we know that

$$\int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\int_Q \left\|\frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s\partial t}\right\|^2 dsdt\right\} d\vec{x} = \rho^{\nu l_n m_n}. \tag{2.3}$$

Let

(2.4)

$$G_{n,q} = \rho^{-\nu l_n m_n}\int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\int_Q \left\|\frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s\partial t}\right\|^2 dsdt\right\} F_q(\vec{x})\, d\vec{x}$$

and

$$(2.5)\quad H_n = \rho^{-\nu l_n m_n} \int_{C_2^\nu} \exp\left\{ \frac{\rho^2-1}{2\rho^2} \int_Q \left\| \frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s \partial t} \right\|^2 ds dt \right\} F(\vec{x})\, d\vec{x}$$

Then from (2.3), (2.4), (2.5) and the dominated conbvergence theorem, we have that

$$(2.6)\qquad \lim_{q\to\infty} G_{n,q} = H_n$$

for $n = 1, 2, \cdots$, and that

$$(2.7)$$
$$\begin{aligned} &|G_{n,q} - H_n| \\ &\le \rho^{-\nu l_n m_n} \int_{C_2^\nu} \exp\left\{ \frac{\rho^2-1}{2\rho^2} \int_Q \left\| \frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s \partial t} \right\|^2 ds dt \right\} |F_q(\vec{x}) - F(\vec{x})| d\vec{x} \\ &\le \sup_{x\in\Omega} |F_q(\vec{x}) - F(\vec{x})|. \end{aligned}$$

Thus (2.6) holds uniformly in n for all positive integers n, and hence for $q = 1, 2, \cdots$,

$$(2.8)\qquad I_q \equiv \int_{C_2^\nu} F(\rho\vec{x}) d\vec{x} = \lim_{n\to\infty} G_{n,q}$$

and

$$(2.9)\qquad I \equiv \int_{C_2^\nu} F(\rho\vec{x}) d\vec{x} = \lim_{q\to\infty} I_q = \lim_{q\to\infty} \lim_{n\to\infty} G_{n,q}$$

By the iterated limits theorem, it follows from (2.8) and (2.9) that

$$I = \lim_{q\to\infty} \lim_{n\to\infty} G_{n,q} = \lim_{n\to\infty} \lim_{q\to\infty} G_{n,q} = \lim_{n\to\infty} H_n.$$

THEOREM 2.3. *Let $\rho > 0$ and let $\{\sigma_n\}$ be a sequence of subdivions of Q such that the norm $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $l_n m_n$ be the number of sub-rectangles in σ_n. Then if $F \in \bar{S}^u(L_2^\nu(Q))$,*

(2.10)

$$\int_{C_2^\nu} F(\rho\vec{x})d\vec{x}$$
$$= \lim_{n\to\infty} \rho^{-\nu l_n m_n} \int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\int_Q \left\|\frac{\partial^2 \vec{x}_{\sigma_n}(s,t)}{\partial s \partial t}\right\|^2 dsdt\right\} F(\vec{x})d\vec{x}.$$

Proof. By Theorem 1.1 and Proposition 2.2, we have that $S(L_2^\nu(Q)) \subset \Gamma$ and hence that $\bar{S}^u(L_2^\nu)(Q)) \subset \Gamma$.

The following theorem can be obtained by extending Theorem 1.2 in the same way that Theorem 2.3 was obtained by extending Theorem 1.1

THEORM 2.4. *Let $\rho > 0$ and let $\{\psi_n\}$ be a complete orthonormal sequence of functions on Q. Then if $F \in \bar{S}^u(L_2^\nu(Q))$,*

(2.11)

$$\int_{C_2^\nu} F(\rho\vec{x})c\vec{x}$$
$$= \lim_{n\to\infty} \rho^{-\nu n} \int_{C_2^\nu} \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{\nu}\sum_{j=1}^{n}\left[\int_Q \psi_j(s,t)\widetilde{dx}_k(s,t)\right]^2\right\} F(\vec{x})d\vec{x}.$$

REMARK 2.5. Using extensions of the techniques developed in this paper, we can formulate the counterparts for N-parameter Wiener space.

EXAMPLE. We let $\nu = 1$ and $[a,b] = [c,d] = [0,\pi]$ and define $\psi_j(s) = \sin js$ for $j = 1,2,\cdots$. Then $\{\psi_{j,k}(s,t) = \psi_j(s)\psi_k(t)\}$ is a complete orthonormal set on $Q = [0,\pi]\times[0,\pi]$. Define the functional $F : C_2(Q) \longrightarrow C$ by

$$F(x) = \exp\{\alpha \int_Q x(s,t)\cos s \cos t dsdt\}$$

for $x \in C_2(Q)$ where α is a real or complex number. We now use the $\{\psi_{j,k}\}$ and F, and evaluate the two sides of (2.11). By using integration by parts formula and the Paley-Wiener-Zygmund theorem, we obtain

$$\int_{C_2} F(\rho x)dx = \int_{C_2} \exp\{\frac{\alpha\rho\pi}{2}\int_Q \psi_{1,1}(s,t)dx(s,t)\}dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\{\frac{\alpha\rho\pi}{2}u\}\exp\{-u^2/2\}du = \exp\{(\alpha\rho\pi)^2/8\}.$$

On the other hand, we apply the Paley-Wiener-Zygmund theorem to the right side of (2.11) so that

$$\lim_{l,m\to\infty} \rho^{-lm}\int_{C_2} \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{j=1}^{l}\sum_{k=1}^{m}\left[\int_Q \psi_{j,k}(s,t)\widetilde{dx}(s,t)\right]\right\}^2 F(x)dx$$
$$= \lim_{l,m\to\infty}(2\pi)^{-\frac{lm}{2}}\int_{\mathbb{R}^{lm}} \exp\left\{\frac{\rho^2-1}{2\rho^2}\sum_{j=1}^{l}\sum_{k=1}^{m}u_{j,k}^2\right\}\exp\left\{\frac{\alpha\pi}{2}u_{1,1}\right\}$$
$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}u_{j,k}^2\right\}du_{1,1}\cdots du_{l,m} = \exp\{(\alpha\rho\pi)^2/8\}.$$

Thus we have established that the equation (2.11) is valid for all complex number α. In particular, if α is pure imaginary, then $F(x) \in S(L_2(Q))$. On the other hand, if $Re\alpha \neq 0$, then $F(x)$ is unbounded, so $F(x) \notin \bar{S}^u(L_2(Q))$, and also $F(x) \notin \bar{S}^u(L_2(Q))$. Thus this example shows that the class of functionals for which (2.11) holds is more extensive than $\bar{S}^u(L_2^\nu(Q))$.

References

1. J.M. Ahn K.S. Chang and I Yoo, *Some Banach Algebras of Yeh-Feynman Integrable Functionals*, J. Korean Math. Soc. **24** (1987), 257–266.
2. S. Albeverio and R. Høegh-krohn, *Mathematical Theory of Feynman Path Integrals*, Springer Lecture Notes in Math. 523, 1976.
3. E. Berkson and T.A. Gillespie, *Absolutely Continuous Functions of Two Variables and Well-Bounded Operators*, J. London Math. Soc. **30** (1984), 305 –321.

4. R.H. Cameron and D.A. Storvick, *Some Bananch Algebras of Analytic Feynman Integrable Functionals*, Springer Lecture Notes in Math. 798, 1980.
5. ______, *Change of Scale Formulas for Wiener Integrals*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie 2-Numero **17** (1987), 105–116.
6. ______, *Relationship between the Wiener Integral and the analytic Feynman jIntegral*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie 2-Numero **17** (1987), 117–133.
7. K.S. Chang, *Scale-Invariant Measurability in Yeh-Wiener Spaces*, Korean Math. Soc. **19** (1982), 61–67.
8. K.S. Chang, J.M. Ahn and J.S., *Chang Notes on the Analytic Yeh-Feynman Integrable Functionals*, Rocky Mountain J. Math. **18** (988), 157–165.
9. K.S. Chang G.W. Johnson and D.L. Skoug, *Necessary and Sufficient Conditions for Membership in the Banach Algebra S for Certain Classes of Functions*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Seire 2-Numero **17** (1987), 153–171.
10. R.P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys. **20** (1948), 367–387.
11. E. Hewitt, *Representation ofFunctions as Absouldtely Convergent Fourier-Stieltjes Transforms*, Proc. Amer. Math. Soc. **4** (1953), 663–670.
12. E.W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier Series Vol I*, Dover New York, 1957.
13. G.W. Johnson and D.L. Skoug, *Scale-Invariant Measurability in Wiener Space*, Pacific J. Math. **83** (1979), 157–176.
14. C. Park, *A Generalized Paley-Wiener-Zygmund Integral and its Applications*, Proc. Amer. Math. Soc. **23** (1969), 388-400.
15. Y.H. Park, *Operator Valued function Space Integrals*, Thesis Yonsei Univ., 1988.
16. J. Yeh, *Wiener Measure in a Space of Functions of Two Variables*, Trans. Amer. Math. Soc. **95** (1960), 433–450.
17. J. Yeh, *Stoctastic Processes and the Wiener Integral*, Marcel Dekker New York, 1973.
18. I Yoo, *Sequential Yeh-Feynman Integrals*, Thesis Yonsei Univ. 1986.

Department of Mathematics
Yonsei University
Wonju 222-701, Korea
and
Department of Mathematics
Mokpo National University
Mokpo 534-729, Korea